MARTIN BOUNDARY OF A REFLECTED RANDOM WALK ON A HALF-SPACE

IRINA IGNATIOUK-ROBERT

Abstract. The complete representation of the Martin compactification for reflected random walks on a half-space $\mathbb{Z}^d \times \mathbb{N}$ is obtained. It is shown that the full Martin compactification is in general not homeomorphic to the "radial" compactification obtained by Ney and Spitzer for the homogeneous random walks in $\mathbb{Z}^d$: convergence of a sequence of points $z_n \in \mathbb{Z}^{d-1} \times \mathbb{N}$ to a point of on the Martin boundary does not imply convergence of the sequence $z_n/|z_n|$ on the unit sphere S^d. Our approach relies on the large deviation properties of the scaled processes and uses Pascal's method combined with the ratio limit theorem. The existence of non-radial limits is related to non-linear optimal large deviation trajectories.

1. Introduction and main results

For an irreducible transient Markov chain $(Z(t))$ on a countable set E having Green's function $G(z, z')$, the Martin compactification $\tilde{E}$ is the smallest compactification of the set E for which the Martin kernels

$$K(z, z') = G(z, z')/G(z_0, z')$$

extend continuously with respect to the second variable z' for every $z \in E$. A point $\eta \in \partial E = \tilde{E} \setminus E$ is said to belong to the minimal Martin boundary if $K(\cdot, \eta)$ is a minimal harmonic function (see Woess [23] for the precise definitions). An explicit representation of the Martin boundary and the minimal Martin boundary $\partial_M E \subset \partial E$ allows to describe all harmonic functions of the Markov chain $(Z(t))$: by Poisson-Martin representation theorem, every positive harmonic function h is of the form

$$h(z) = \int_{\partial_M E} K(z, \eta)\, d\nu(\eta)$$

where ν is a positive Borel measure on $\partial_M E$. Moreover, by convergence theorem, for every $z \in E$, the sequence $Z(n)$ converges P_z almost surely to a random variable taking the values in $\partial_M E$.

An explicit description of the Martin boundary is usually a non-trivial problem. The most of the existing results in this domain were obtained for the homogeneous processes (see Woess [23] and the references therein). One of the few results where the full Martin compactification was obtained for non-homogeneous processes is the paper of Kurkova and Malyshev [17]. They considered random walks on a half-plane $\mathbb{Z} \times \mathbb{N}$ and in the quadrant $\mathbb{Z}^2_+ = \mathbb{N} \times \mathbb{N}$ which behave as a homogeneous nearest neighbors random walk in the interior of the domain and have some different

Date: June 5, 2009.
1991 *Mathematics Subject Classification.* Primary 60J50; Secondary 60F10; 60J45.
Key words and phrases. Martin boundary.Sample path large deviations. Random walk.

(homogeneous) transition probabilities on the boundary. Their results show a very surprising relationship between the Martin compactification and the optimal large deviation trajectories described for such processes obtained in Ignatyuk, Malyshev and Scherbakov [16]. Let us illustrate this relationship on the example of the reflected random walks on the half-plane : the results of Kurkova and Malyshev show that for such a random walk, there are two real values $0 \leq \theta_1 \leq \theta_2 \leq \pi$ such that

i) a sequence of points $z_n \in \mathbb{Z}\times\mathbb{N}$ with $\lim |z_n| = \infty$ converges to a point $\eta(\theta)$ of the Martin boundary if the sequence $z_n/|z_n|$ converges to a point $e^{i\theta}$ on $S^2_+ = \{e^{i\theta} : \theta\in[0,\pi]\}$;
ii) two sequences $z_n, z'_n \in \mathbb{Z}\times\mathbb{N}$ with $\lim |z_n| = \lim |z'_n| = \infty$, $\lim z_n/|z_n| = e^{i\theta}$ and $\lim z'_n/|z'_n| = e^{i\theta'}$ converge to the same point $\eta(\theta) = \eta(\theta')$ of the Martin boundary if an only if
   – either $\theta = \theta' \in [\theta_1, \theta_2], \quad \mod (2\pi)$,
   – or $\theta, \theta' \in [0, \theta_1], \quad \mod (2\pi)$,
   – or $\theta, \theta' \in [\theta_2, \pi], \quad \mod (2\pi)$.

In [16] it was shown that for every $T > 0$, the family of scaled random walks

$$(Z^\varepsilon(t) \doteq \varepsilon Z([t/\varepsilon]),\, t\in[0,T])$$

satisfy sample path large deviation principle with a rate function $I_{[0,T]}(\phi)$ and that with the same values θ_1 and θ_2, the following assertions hold.

- For $\theta \in [\theta_1, \theta_2]$, the optimal large deviation trajectory $\phi_\theta : [0, T_\theta] \to \mathbb{R}\times\mathbb{R}_+$ minimizing the rate function $I_{[0,T]}(\phi)$ over all $T > 0$ and all continuous functions $\phi : [0, T] \to \mathbb{R}\times\mathbb{R}_+$ with given $\phi(0) = 0$ and $\phi(T) = e^{i\theta} \in S^2_+$ is linear : $\phi_\theta(t) = e^{i\theta} t/T_\theta$ with some $T_\theta > 0$;
- while for $\theta \in [0, \theta_1] \cup [\theta_2, \pi]$, such a trajectory is piece-wise linear and is of the form

$$\phi_\theta(t) \;=\; \begin{cases} \gamma_\theta t/T'_\theta & \text{for } t\in[0,T'_\theta] \\ \gamma_\theta + (e^{i\theta}-\gamma_\theta)(t-T'_\theta)/(T_\theta - T'_\theta) & \text{for } t \in [T'_\theta, T_\theta] \end{cases}$$

  with some $T_\theta > T_\theta > 0$ where γ_θ is a unique point on the boundary $\mathbb{R}\times\{0\}$ for which

$$\arg(e^{i\theta} - \gamma_\theta) = \begin{cases} \theta_1 & \text{if } \theta\in[0,\theta_1], \\ \theta_2 & \text{if } \theta \in [\theta_2, \pi]. \end{cases}$$

Unfortunately, the method proposed by Kurkova and Malyshev [17] required very particular properties of the process : they considered the random walks for which the only non-zero transitions in the interior of the domain are on the nearest neighbors: $p(z, z\pm e_i) = \mu(\pm e_i)$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. For such random walks, the jump generating function is defined by

$$\varphi(x,y) \;=\; \mu(e_1)x + \mu(-e_1)x^{-1} + \mu(e_2)y + \mu(-e_2)y^{-1}$$

and the equation $xy(1-\varphi(x, y)) = 0$ determines an elliptic curve $\mathbf{S}$ which is homeomorphic to the torus. To identify the Martin boundary, a functional equation was derived for the generating function of the Green's function and the asymptotics of the Green's function were calculated by using the methods of complex analysis on the elliptic curve $\mathbf{S}$. Such a method seems to be unlikely to apply in a more general situation, for higher dimensions or when the jump sizes are arbitrary, because the proof is based on the geometrical properties of the elliptic curve $\mathbf{S}$: even for

the 2-dimensional case, if a random walk has an additional non-zero transition $p(z, z+u) = \mu(u)$ with $|u| > 2$, the equation $xy(1-\varphi(x,y)) = 0$ is not of the second order and consequently, the corresponding elliptic curve is not homeomorphic to the torus.

Since the large deviation methods extend easily for an arbitrary dimension and for arbitrary jumps, a natural idea is to use them in order to identify the Martin boundary. The similarities of the results of Kurkova and Malyshev [17] and the large deviation results of Ignatyuk, Malyshev and Scherbakov [16] suggest that such an approach should be possible. The first result in this domain was obtained in Ignatiouk-Robert [15] for a homogeneous random walk $(Z_+(t))$ on $\mathbb{Z}^d$ killed upon hitting the negative half-space $\mathbb{Z}^{d-1}\times(-\mathbb{N})$: the large deviation technique was combined there with Bernoulli part decomposition due to Foley and McDonald [6]. The main steps of this method can be summarized as follows :

– The first step is a ratio limit theorem: Bernoulli part decomposition was used to identify the limits of the Martin kernel $K(z, z_n)$ when the logarithmic asymptotic of Green's function for a given sequence (z_n) is zero.
– The logarithmic asymptotics of Green's function were obtained with the large deviation technique.
– An appropriated exponential change of the measure was finally used in order to apply the ratio limit theorem for a twisted Markov process for which the corresponding logarithmic asymptotic of Green's function is zero.

In the present paper the large deviation method is developed in order to identify the Martin boundary for a reflected random walk $(Z(t))$ on the half-space $\mathbb{Z}^{d-1}\times\mathbb{N}$. Such a random walk behaves as a homogeneous random walk in the interior of the half-space and has some different transition probabilities on the boundary hyper-plane $\mathbb{Z}^{d-1}\times\{0\}$. Here, the approach of Ignatiouk-Robert [15] is not only harder to apply but also it does not work in general because the corresponding twisted process does not exist. To solve this problem we refine the large deviation technique.

We show that the family of scaled processes $(Z^\varepsilon(t) = \varepsilon Z([t/\varepsilon]), t\in[0,T])$ satisfies sample path large deviation principle with a good rate function $I_{[0,T]}$ and that the logarithmic asymptotics of Green's function $G(z, z_n)$ of the original process $(Z(t))$ when $|z_n|\to\infty$ and $z_n/|z_n|\to q$ are determined by the quasi-potential

$$I(0,q) \;=\; \inf_{T>0}\; \inf_{\phi:\ \phi(0)=0,\ \phi(T)=q} I_{[0,T]}(\phi)$$

which represents an optimal large deviation cost to go from the point 0 to the point q. Next, the method of [15] is used to identify the limit of the Martin kernel $K(z, z_n)$ when $|z_n|\to\infty$ and the limit $z_n/|z_n|\to q$ belongs to the boundary hyper-plane $\mathbb{R}^{d-1}\times\{0\}$. This is the first step of our proof.

For $q\not\in\mathbb{R}^{d-1}\times\{0\}$ we consider a function $\phi : [0,T]\to\mathbb{R}^d$ with $\phi(0) = 0$ and $\phi(T) = q$ where the minimum $I(0,q)$ is achieved. Such a function ϕ represents an optimal large deviation path from 0 to q. It is shown that every optimal large deviation path from 0 to q leaves the boundary hyper-plane $\mathbb{R}^{d-1}\times\{0\}$ at some point $\gamma_q\in\mathbb{R}^{d-1}\times\{0\}$ and that Green's function $G(z, z_n)$ can be decomposed into a main part determined by γ_q and the corresponding negligible part. The main part of $G(z, z_n)$ corresponds to the trajectories of the process $(Z(t))$ that leave the boundary hyper-plane in a $\delta|z_n|$-neighborhood of the point $\gamma_q|z_n|$. With this approach we identify the limit of the Martin kernel $K(z, z_n)$ when $|z_n|\to\infty$ and $z_n/|z_n|\to q$ for any $q\in\mathbb{R}^{d-1}\times[0,+\infty[$.

The reflection on the boundary is not only harder to tackle but also yields very different and interesting results. Contrary to the case analyzed in [15], here the convergence to the Martin boundary can be non-radial : a convergence to a point on the Martin boundary of a sequence (z_n) does not imply the convergence of the sequence $z_n/|z_n|$ on the unit sphere. We obtain this result as a consequence of the existence of non-linear optimal large deviation trajectories.

1.1. **Main result.** We consider a Markov process $Z(t) = (X(t), Y(t))$ on $\mathbb{Z}^{d-1}\times\mathbb{N}$ with transition probabilities

$$p(z,z') \;=\; \begin{cases} \mu(z'-z) & \text{for } z=(x,y), z'\in\mathbb{Z}^{d-1}\times\mathbb{N} \text{ with } y>0,\\ \mu_0(z'-z) & \text{for } z=(x,y), z'\in\mathbb{Z}^{d-1}\times\mathbb{N} \text{ with } y=0\end{cases} \tag{1.1}$$

where μ and μ_0 are two different probability measures on $\mathbb{Z}^d$ having the means

$$m \doteq \sum_{z\in\mathbb{Z}^d} z\mu(z) \qquad \text{and} \qquad m_0 \doteq \sum_{z\in\mathbb{Z}^d} z\mu_0(z). \tag{1.2}$$

Throughout this paper we denote by $\mathbb{N}$ the set of all non-negative integers : $\mathbb{N} = \{0,1,2,\ldots\}$ and we let $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. The assumptions we need on the Markov process $(Z(t))$ are the following.

- (H0) *$\mu(z) = 0$ for $z=(x,y)\in\mathbb{Z}^{d-1}\times\mathbb{Z}$ with $y<-1$ and $\mu_0(z) = 0$ for $z=(x,y)\in\mathbb{Z}^{d-1}\times\mathbb{Z}$ with $y<0$.*
- (H1) *The Markov process $Z(t)$ is irreducible on $\mathbb{Z}^{d-1}\times\mathbb{N}$.*
- (H2) *The homogeneous random walk $S(t)$ on $\mathbb{Z}^d$ having transition probabilities $p_S(z,z') = \mu(z'-z)$ is irreducible on $\mathbb{Z}^d$ and the last coordinate of $S(t)$ is an aperiodic random walk on $\mathbb{Z}$.*
- (H3)
$$m\neq 0 \qquad \text{and} \qquad \frac{m}{|m|}+\frac{m_0}{|m_0|}\neq 0.$$
- (H4) *The jump generating functions*
$$\varphi(a) \;=\; \sum_{z\in\mathbb{Z}^d}\mu(z)e^{a\cdot z} \quad \textit{and} \quad \varphi_0(a) \;=\; \sum_{z\in\mathbb{Z}^d}\mu_0(z)e^{a\cdot z} \tag{1.3}$$
are finite everywhere on $\mathbb{R}^d$.

Under the above assumptions, the sets

$$D \doteq \{a\in\mathbb{R}^d : \varphi(a)\le 1\} \quad \text{and} \quad D_0 \doteq \{a\in\mathbb{R}^d : \varphi_0(a)\le 1\} \tag{1.4}$$

are convex and the set D is moreover compact (see [12]). The following parts of the boundary ∂D are important for our analysis :

$$\partial_0 D \doteq \{a\in\partial D:\ \nabla\varphi(a)\in\mathbb{R}^{d-1}\times\{0\}\}$$
$$\partial_+ D \doteq \{a\in\partial D:\ \nabla\varphi(a)\in\mathbb{R}^{d-1}\times[0,+\infty[\}$$

and

$$\partial_- D \doteq \{a\in\partial D:\ \nabla\varphi(a)\in\mathbb{R}^{d-1}\times]-\infty,0]\}.$$

For $a\in D$, denote by $\overline{a}$ the unique point on the boundary $\partial_- D$ which has the same first $(d-1)$ coordinates as the point a and let

$$\hat{D} = \{a\in D : \varphi_0(\overline{a})\le 1\}. \tag{1.5}$$

Remark that under the hypotheses (H0)-(H1), for any $a\in D$,

$$\varphi_0(\overline{a})\le\varphi_0(a)$$

because the function $a = (\alpha,\beta) \to \varphi_0(a)$ is increasing with respect to the last coordinate β of $a = (\alpha,\beta) \in \mathbb{R}^d$. This inequality implies another useful representation of the set $\hat{D}$:

$a = (\alpha,\beta) \in \hat{D}$ if and only if $a \in D$ and $a' = (\alpha,\beta') \in D \cap D_0$ for some $\beta' \in \mathbb{R}$

or equivalently,

$$\hat{D} \;=\; (\Theta \times \mathbb{R}) \cap D \tag{1.6}$$

where

$$\Theta \doteq \{\alpha \in \mathbb{R}^{d-1} : \inf_{\beta\in\mathbb{R}} \max\{\varphi(\alpha,\beta), \varphi_0(\alpha,\beta)\} \le 1\}. \tag{1.7}$$

The set $\Theta \times \{0\}$ is therefore the orthogonal projection of the set $D \cap D_0$ onto the hyper-plane $\mathbb{R}^{d-1} \times \{0\}$. Remark finally that $\partial_0 D = \partial_+ D \cap \partial_- D$ and for $a \in \partial_+ D$, $a = \overline{a}$ if and only if $a \in \partial_0 D$.

It is moreover convenient to introduce the following notations : for $a \in \hat{D} = (\Theta \times \mathbb{R}) \cap D$, we denote by $V(a)$ the normal cone to the set $\hat{D}$ at the point a and for $a \in \hat{D} \cap \partial_+ D = (\Theta \times \mathbb{R}) \cap \partial_+ D$ we define the function h_a on $\mathbb{Z}^{d-1} \times \mathbb{N}$ by letting

$$h_a(z) \;=\; \begin{cases} e^{a\cdot z} - \dfrac{1-\varphi_0(a)}{1-\varphi_0(\overline{a})}\, e^{\overline{a}\cdot z} & \text{if } \; a \notin \partial_0 D \; \text{ and } \; \varphi_0(\overline{a}) < 1, \\[2ex] y e^{a\cdot z} + \dfrac{\frac{\partial}{\partial\beta}\varphi_0(a)}{(1-\varphi_0(a))}\, e^{a\cdot z} & \text{if } \; a = \overline{a} \in \partial_0 D \; \text{ and } \; \varphi_0(a) < 1, \\[2ex] e^{\overline{a}\cdot z} & \text{if } \; \varphi_0(\overline{a}) = 1 \end{cases} \tag{1.8}$$

where $\frac{\partial}{\partial\beta}\varphi(a)$ denotes the partial derivative of the function $a \to \varphi(a)$ with respect to the last coordinate $\beta \in \mathbb{R}$ of $a = (\alpha,\beta)$.

We denote by $\mathcal{S}^d_+$ a half-sphere $\mathcal{S}^d \cap \mathbb{R}^{d-1} \times \mathbb{R}_+$ and $G(z,z')$ denotes Green's function of the Markov process $(Z(t))$.

Our preliminary results show that for any $q \in \mathcal{S}^d_+$, there is a unique point $\hat{a}(q) \in \hat{D} \cap \partial_+ D$ for which $q \in V(\hat{a}(q))$ and that for every $a \in \hat{D} \cap \partial_+ D$,

$$V(a) \;=\; \begin{cases} \big\{c\nabla\varphi(a) : c \ge 0\big\} & \text{if either } \varphi_0(\overline{a}) < 1 \\ & \text{or } a = \overline{a} \in \partial_0 D, \\ \big\{c_1\nabla\varphi(a) + c_2(\nabla\varphi_0(\overline{a}) + \kappa_a \nabla\varphi(\overline{a})) : \; c_i \ge 0\big\} & \text{if } \varphi_0(\overline{a}) = 1 \\ & \text{and } a \notin \partial_0 D \end{cases} \tag{1.9}$$

where

$$\kappa_a = -\left.\frac{\partial\varphi_0(\alpha,\beta)}{\partial\beta}\left(\frac{\partial\varphi(\alpha,\beta)}{\partial\beta}\right)^{-1}\right|_{(\alpha,\beta)=\overline{a}}$$

(see Lemma 2.3 and Lemma 2.5 below).

The main result of our paper is the following theorem.

Theorem 1. *Under the hypotheses (H0)-(H4), the following assertions hold :*

(i) *the Markov process $Z(t)$ is transient;*

(ii) *for any* $a \in \hat{D} \cap \partial_+ D$ *and any sequence of points* $z_n \in \mathbb{Z}^{d-1} \times \mathbb{N}$ *with* $\lim_{n\to\infty} |z_n| = \infty$,

$$\lim_{n\to\infty} G(z, z_n)/G(z_0, z_n) = h_a(z)/h_a(z_0), \qquad \forall\, z \in \mathbb{Z}^{d-1} \times \mathbb{N} \tag{1.10}$$

when $\lim_{n\to\infty} \mathrm{dist}(V(a), z_n/|z_n|) = 0$.

Assertion (ii) proves that a sequence $z_n \in \mathbb{Z}^{d-1} \times \mathbb{N}$ with $\lim_{n\to\infty} |z_n| = \infty$, converges to a point on the Martin boundary if and only if

$$\lim_{n\to\infty} \; \mathrm{dist}\,(V(a), z_n/|z_n|) = 0$$

for some $a \in \hat{D} \cap \partial_+ D$. Recall that for a homogeneous random walk on $\mathbb{Z}^d$ (see Ney and Spitzer [18]), a sequence $z_n \in \mathbb{Z}^d$ converges to a point of the Martin boundary if and only if $\lim_{n\to\infty} |z_n| = \infty$ and the sequence $z_n/|z_n|$ converges to a point on the unit sphere S^d. For the reflected random walk on the half-space $\mathbb{Z}^{d-1} \times \mathbb{N}$, Theorem 1 provides the existence of non-radial limits : if the mapping $\hat{a} : \mathcal{S}^d_+ \to \hat{D} \cap \partial_+ D$ is not one to one then the convergence to a point on the Martin boundary does not imply convergence of the sequence $z_n/|z_n|$. The explicit representation (1.9) of the normal cone $V(a)$ shows that such a mapping is not one to one in a quite general situation : when $\varphi_0(\overline{a}) = 1$ for some $\overline{a} \in \partial_- D$.

1.2. **The overview of the proof.** To prove Theorem 1 we identify first the harmonic functions of the process $(Z(t))$. Since the transition probabilities of the Markov process $(Z(t))$ are invariant with respect to the translations on $z \in \mathbb{Z}^{d-1} \times \{0\}$ and since the Markov process $(Z(t))$ is irreducible then the same arguments as in Doob, Snell and Williamson [7] (see the proof of Theorem 5) show that every minimal harmonic function is of the form

$$h(x, y) = \exp(\alpha \cdot x) h(0, y), \qquad \forall (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}$$

with some $\alpha \in \mathbb{R}^{d-1}$. We prove that the constant multiples of the functions h_a with $a = (\alpha, \beta) \in \hat{D} \cap \partial_+ D$, are the only minimal non-negative harmonic functions of the Markov process $(Z(t))$. These arguments prove the first assertion of Theorem 1 because under our hypotheses, $\{0\} \subset \hat{D} \cap \partial_+ D \neq \{0\}$.

To prove the assertion (ii), we identify first the logarithmic asymptotics of Green's function, by using the large deviation method. The results of Dupuis, Ellis and Weiss [8], Dupuis and Ellis [10] and Ignatiouk [13, 14] are used to show that the family of scaled processes $(Z^\varepsilon(t) = \varepsilon Z([t/\varepsilon]),\, t \in [0, T])$ satisfies sample path large deviation principle with a good rate function $I_{[0,T]}(\phi)$ having an explicit form. The quasi-potential $I(0, q)$ of the rate function $I_{[0,T]}(\phi)$ represents an optimal large deviation cost to go from the point 0 to the point q :

$$I(0, q) \;=\; \inf_{T>0} \; \inf_{\phi:\phi(0)=0,\, \phi(T)=q} I_{[0,T]}(\phi)$$

We show that for any $q \in \mathbb{R}^{d-1} \times \mathbb{R}_+$ and any sequence of points $z_n \in \mathbb{Z}^{d-1} \times \mathbb{N}$ with $\lim |z_n| = +\infty$ and $\lim z_n/|z_n| = q$, the following equalities hold

$$\lim_{n\to\infty} \frac{1}{|z_n|} \log G(z, z_n) \;=\; -I(0, q) \;=\; \sup_{a\in\hat{D}} \; a \cdot q \;=\; \hat{a}(q) \cdot q, \qquad \forall z \in \mathbb{Z}^{d-1} \times \mathbb{N}. \tag{1.11}$$

When $\lim_{n\to\infty} z_n/|z_n| = q \in \mathbb{R}^{d-1} \times \{0\}$ and $\overline{\hat{a}(q)} = 0$, the proof of (1.10) uses the following arguments :

– from (1.11) we obtain the equality

$$\lim_{n\to\infty} \frac{1}{n} \log G\big(z_0, z_n\big) \ = \ 0$$

– and next, using the ratio limit theorem of [15] we get (1.10).

To get (1.10) for a sequence $z_n \in \mathbb{Z}^{d-1}\times\mathbb{N}$ with $\lim_{n\to\infty} z_n/|z_n| = q \in \mathbb{R}^{d-1}\times\{0\}$ and $\overline{\hat{a}(q)} \neq 0$, the above arguments are combined together with the exponential change of measure : the ratio limit theorem is applied for a sub-stochastic twisted Markov chain having transition probabilities $\tilde{p}(z,z') \ = \ p(z,z')\exp(a\cdot(z'-z))$ with a parameter $a = \overline{\hat{a}(q)}$.

Similar arguments are used in order to prove (1.10) for a sequence of points $z_n \in \mathbb{Z}^{d-1}\times\mathbb{N}$ with $\lim_{n\to\infty} z_n/|z_n| = q \in \mathbb{R}^{d-1}\times\mathbb{R}^*_+$ when $\varphi_0\left(\overline{\hat{a}(q)}\right) < 1$. The only difference is here that there is no suitable exponential change of measure. Instead of the exponential change of measure we consider a twisted Markov chain $(\tilde{Z}(t))$ with transition probabilities $\tilde{p}(z,z') \ = \ p(z,z')h_{\hat{a}(q)}(z')/h_{\hat{a}(q)}(z)$. For the twisted Green's function $\tilde{G}(z,z') = G(z,z')h_{\hat{a}(q)}(z')/h_{\hat{a}(q)}(z)$, the equality

$$\lim_{n\to\infty} \frac{1}{n} \log \tilde{G}\big(z_0, z_n\big) \ = \ 0$$

follows from the relations (1.11) and the explicit form of the harmonic function $h_{\hat{a}(q)}$.

The case when $\lim_{n\to\infty} z_n/n \ = \ q \ \in \ \mathbb{R}^{d-1}\times\mathbb{R}^*_+$ and $\varphi_0\left(\overline{\hat{a}(q)}\right) \ = \ 1$ is more difficult. In this case, we can not use the above arguments because there is no harmonic functions h satisfying the equality

$$\lim_{n\to\infty} \frac{1}{n} \log\big(G\big(z_0, z_n\big)h(z_n)\big) \ = \ 0.$$

Instead, we use Pascal's method combined with the renewal equation

$$G(z,z_n) \ = \ G_+(z,z_n) \ + \sum_{\substack{w\in E\setminus\mathbb{Z}^{d-1}\times\{0\},\\ w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*}} G(z,w)p(w,w')G_+(w',z_n). \tag{1.12}$$

$G_+(z,z')$ denotes here the mean number of visits of the point z' starting from z before hitting the boundary hyperplane $\mathbb{Z}^{d-1}\times\{0\}$. Here, the main ideas of our proof are the following :

For every point $q\in\mathbb{R}^{d-1}\times\mathbb{R}^*_+$, the normal cone $V(\hat{a}(q))$ is generated by the vectors γ_q and $q-\gamma_q$ with some uniquely defined $\gamma_q\in\mathbb{R}^{d-1}\times\{0\}$. In the large deviation scaling, the point γ_q corresponds to an optimal way from 0 to q. Because of the influence of the boundary, the optimal ways are not linear, an optimal way from 0 to $q\in\mathbb{R}^{d-1}\times\mathbb{R}^*_+$ follows first a linear trajectory on the boundary hyperplane $\mathbb{R}^{d-1}\times\{0\}$ before hitting the point $\gamma_q\in\mathbb{R}^{d-1}\times\{0\}$ and next follows another linear trajectory from γ_q to q in the interior of the half-space.

The right hand side of the renewal equation (1.12) is decomposed into a principal part

$$\Xi^q_\delta(z,z_n) \ = \ G_+(z,z_n)1_{\{\gamma_q=0\}} + \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\}:|w-\gamma_q|z_n||<\delta|z_n|,\\ w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*}} G(z,w)p(w,w')G_+(w',z_n)$$

corresponding to the optimal large deviation way to go from 0 to q, and the negligible part

$$G_+(z,z_n)1_{\{\gamma_q\neq 0\}} \;+ \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\}:|w-\gamma_q|z_n||\geq\delta|z_n|,\\ w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*}} G(z,w)p(w,w')G_+(w',z_n).$$

Next, for those $q\in\mathbb{R}^{d-1}\times\mathbb{R}^*$ for which $\gamma_q=0$, we obtain (1.10) by using the results of [15]. When $\gamma_q\neq 0$, the equality (1.10) is obtained from the convergence

$$G(z,w)/G(z_0,w) \;\to\; h_{\hat a(\gamma_q)}(z)/h_{\hat a(\gamma_q)}(z_0)$$

as $|w|\to\infty$ and $w/|w|\to\gamma_q$. We use here the fact that $\hat a(\gamma_q)=\hat a(q)$ and that $\gamma_q\in\mathbb{R}^{d-1}\times\{0\}$ (recall that for $q\in\mathbb{R}^{d-1}\times\{0\}$, the equality (1.10) is proved by using the ratio limit theorem).

Our paper is organized as follows. Section 2 is devoted to the preliminary results. The harmonic functions of the Markov process $(Z(t))$ are identified in Section 3. In Section 4 we prove that our Markov process satisfies strong communication condition. This property is needed to establish sample path large deviation principle for the family of scaled processes and also to apply the ratio limit theorem. Section 5 is devoted to large deviation results. In Section 6 we apply large deviation results to decompose the right hand side of the renewal equation (1.12) into a principal part and a negligible part. Section 7 is devoted to the ratio limit theorem. The proof Theorem 1 is given in Section 8.

2. Preliminary results

Let $\tau=\inf\{t\geq 1 \;:\; Z(t)\in\mathbb{Z}^{d-1}\times\{0\}\}$ denote the first time when the process $Z(t)$ returns to the boundary hyper-plane $\mathbb{Z}^{d-1}\times\{0\}$. Recall that for $z,z'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*$,

$$G_+(z,z') \;=\; \sum_{t=0}^{\infty}\mathbb{P}_z(Z(t)=z',\,\tau>t)$$

is Green's function of a homogeneous random walk $Z_+(t)$ on $\mathbb{Z}^{d-1}\times\mathbb{N}^*$ having transition probabilities $p_S(z,z')=\mu(z'-z)$ and killed upon hitting the half-space $\mathbb{Z}^{d-1}\times(-\mathbb{N})$. The homogeneous random walk on $\mathbb{Z}^d$ having transition probabilities $p_S(z,z')=\mu(z'-z)$, $z,z'\in\mathbb{Z}^d$, and its Green's function are denoted by $S(t)$ and $G_S(z,z')$ respectively. On several occasions we will need the following relations.

Lemma 2.1. *Under the hypotheses (H1) and (H2), for any $a\in D$*

$$G_+(z,z') \;\leq\; \exp(a\cdot(z-z'))G_S(0,0) \qquad \forall z,z'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*.$$

If moreover $\varphi_0(a)\leq 1$ then also

$$G(z,z') \;\leq\; \exp(a\cdot(z-z'))G(z',z') \qquad \forall z,z'\in\mathbb{Z}^{d-1}\times\mathbb{N}.$$

Proof. Indeed, for $a\in D$, the exponential function $z\to\exp(a\cdot z)$ is super-harmonic for the Markov process $Z_+(t)$. By Harnack's inequality from this it follows that

$$\begin{aligned}
G_+(z,z')/G_S(0,0) \;&=\; G_+(z,z')/G_S(z',z')\\
&\leq\; G_+(z,z')/G_+(z',z') \;=\; \mathbb{P}_z(Z_+(t)=z' \text{ for some } t\in\mathbb{N})\\
&\leq\; \exp(a\cdot(z-z'))
\end{aligned}$$

for all $z, z' \in \mathbb{Z}^{d-1} \times \mathbb{N}^*$. Moreover, for those $a \in D$ for which $\varphi_0(a) \le 1$, the exponential function $z \to \exp(a \cdot z)$ is also super-harmonic for the Markov process $Z(t)$. Hence, using again Harnack's inequality we obtain

$$G(z,z')/G(z',z') \;=\; \mathbb{P}_z(Z(t) = z' \text{ for some } t \in \mathbb{N}) \;\le\; \exp(a \cdot (z - z'))$$

for all $z, z' \in \mathbb{Z}^{d-1} \times \mathbb{N}$. □

Lemma 2.2. *Under the hypotheses (H0) and (H1), for any $a \in D$*

$$\mathbb{E}_z\big(\exp(a \cdot Z(\tau));\, \tau < \infty\big) \;=\; \begin{cases} \exp(\overline{a} \cdot z) & \text{if } z \in \mathbb{Z}^{d-1} \times \mathbb{N}^*, \\ \exp(\overline{a} \cdot z)\varphi_0(\overline{a}) & \text{if } z \in \mathbb{Z}^{d-1} \times \{0\}. \end{cases}$$

Proof. Indeed, since $Z(\tau) \in \mathbb{Z}^{d-1} \times \{0\}$ then for any $a \in D$ and $z \in \mathbb{Z}^{d-1} \times \mathbb{N}$

$$\mathbb{E}_z\big(\exp(a \cdot Z(\tau) - \overline{a} \cdot z);\, \tau < \infty\big) \;=\; \mathbb{E}_z\big(\exp(\overline{a} \cdot (Z(\tau) - z));\, \tau < \infty\big)$$

because according to the definition of the mapping $a \to \overline{a} \in \partial_- D$ (see Section 1), $a \cdot z = \overline{a} \cdot z$ for all $z \in \mathbb{Z}^{d-1} \times \{0\}$. For $z \in \mathbb{Z}^{d-1} \times \mathbb{N}^*$, the right hand side of this equality is equal to the probability that a twisted homogeneous random walk $\tilde{Z}(t)$ on $\mathbb{Z}^d$ with transition probabilities $\tilde{p}(z,z') \;=\; \exp(\overline{a} \cdot (z' - z))\mu(z' - z)$ starting from z ever hits the hyper-plane $\mathbb{Z}^{d-1} \times \{0\}$. Such a twisted random walk has a finite variance (this is a consequence of the assumption (H4)) and mean

$$\mathbb{E}_z(\tilde{Z}(1) - z) \;=\; \sum_{z' \in \mathbb{Z}^d} (z' - z) \exp(\overline{a} \cdot (z' - z))\mu(z' - z) \;=\; \nabla\varphi(\overline{a}).$$

The last coordinate of $\nabla\varphi(\overline{a})$ is negative or zero because $\overline{a} \in \partial_- D$. Since $\mu(z) = 0$ for all $z = (x,y)$ with $y < -1$, the twisted random walk $\tilde{Z}(t)$ starting at any point $z \in \mathbb{Z}^{d-1} \times \mathbb{N}^*$ hits the hyper-plane $\mathbb{Z}^{d-1} \times \{0\}$ with probability 1 and consequently,

$$\mathbb{E}_z\big(\exp(a \cdot Z(\tau));\; \tau < \infty\big) \;=\; \mathbb{E}_z\big(\exp(\overline{a} \cdot Z(\tau));\; \tau < \infty\big) \;=\; \exp(\overline{a} \cdot z) \tag{2.1}$$

for every $z \in \mathbb{Z}^{d-1} \times \mathbb{N}^*$. Finally, by Markov property, for $z \in \mathbb{Z}^{d-1} \times \{0\}$ we get

$$\begin{aligned} \mathbb{E}_z(\exp(a \cdot Z(\tau));\; \tau < \infty) = &\sum_{z' \in \mathbb{Z}^{d-1} \times \mathbb{N}^*} p(z,z')\mathbb{E}_{z'}(\exp(a \cdot Z(\tau));\; \tau < \infty) \\ &+ \sum_{z' \in \mathbb{Z}^{d-1} \times \{0\}} p(z,z') \exp(a \cdot z') \\ =\; &\varphi_0(\overline{a}) \exp(\overline{a} \cdot z). \end{aligned}$$

The last relation is a consequence of (2.1) and the equality $\overline{a} \cdot z' = a \cdot z'$ for $z' \in \mathbb{Z}^{d-1} \times \{0\}$. □

By strong Markov property Lemma 2.2 implies that

Corollary 2.1. *Under the hypotheses (H0)-(H1), for all $a \in D$, $z \in \mathbb{Z}^{d-1} \times \mathbb{N}$,*

$$\sum_{w \in \mathbb{Z}^{d-1} \times \{0\}} G(z,w) \exp(a \cdot w) \;=\; \begin{cases} (1 - \varphi_0(\overline{a}))^{-1} \exp(\overline{a} \cdot z) & \text{if } \varphi_0(\overline{a}) < 1, \\ +\infty & \text{if } \varphi_0(\overline{a}) \ge 1. \end{cases}$$

The last statement together with Lemma 2.1 implies the following estimate for Green's function.

Corollary 2.2. *Under the hypotheses (H0)-(H2), for any $a \in D$ such that $\varphi(\overline{a}) < 1$, $z \in \mathbb{Z}^{d-1} \times \mathbb{N}$ and $z' \in \mathbb{Z}^{d-1} \times \mathbb{N}^*$,*

$$G(z,z')/G_S(0,0) \;\le\; \exp(a \cdot (z - z')) + \varphi_0(a)(1 - \varphi_0(\overline{a}))^{-1} \exp(\overline{a} \cdot z - a \cdot z') \tag{2.2}$$

Proof. Indeed, let $a \in D$ be such that $\varphi_0(\overline{a}) < 1$. Then for $z \in \mathbb{Z}^{d-1} \times \mathbb{N}$ and $z' \in \mathbb{Z}^{d-1} \times \mathbb{N}^*$, from the renewal equation

$$G(z,z') \;=\; G_+(z,z') \;+\; \sum_{w\in\mathbb{Z}^{d-1}\times\{0\}} \;\sum_{w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*} G(z,w)\mu_0(w,w')G_+(w',z')$$

combined with Lemma 2.1 it follows that

$$\begin{aligned}\frac{G(z,z')}{G_S(0,0)} &\le \exp(a\cdot(z-z')) \;+ \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\},\\ w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*}} G(z,w)\mu_0(w'-w)\exp(a\cdot(w'-z'))\\ &\le \exp(a\cdot(z-z')) \;+\; \varphi_0(a)\sum_{w\in\mathbb{Z}^{d-1}\times\{0\}} G(z,w)\exp(a\cdot(w-z'))\end{aligned}$$

and hence, using by Corollary 2.1 we get (2.2) □

We will need moreover the following consequence of Lemma 2.2.

Corollary 2.3. *Under the hypotheses (H0)-(H4), every point of the set* $\partial_- D\setminus\partial_0 D$ *has a neighborhood in which the function* $a \to \mathbb{E}_0\big(\exp(a\cdot Z(\tau));\, \tau<\infty\big)$ *is finite.*

Now we obtain an explicit representation of the normal cone $V(a)$ for $a \in \hat{D}$. Recall that $V(a)$ is the normal cone to the convex set $\hat{D} = \{a \in D : \varphi_0(\overline{a}) \le 1\} = (\Theta\times\mathbb{R})\cap D$ at the point $a \in \hat{D}$. If the point a belongs to the interior of the set D then clearly $V(a) = \{0\}$. It is sufficient therefore to consider the points on the boundary $\partial\hat{D}$ of $\hat{D}$. According to the definition of the set $\hat{D}$, a point a belongs to the boundary $\partial\hat{D}$ if and only if $\max\{\varphi(a), \varphi_0(\overline{a})\} = 1$.

Lemma 2.3. *Under the hypotheses (H0)-(H4), for every* $a \in \partial\hat{D}$,

(2.3)
$$V(a) \;=\; \begin{cases} \big\{c_1\nabla\varphi(a) + c_2(\nabla\varphi_0(\overline{a}) + \kappa_{\overline{a}}\nabla\varphi(\overline{a})) : c_i \ge 0\big\} & \textit{if } \varphi(a) = \varphi_0(\overline{a}) = 1 \\ & \quad\textit{and } a\notin\partial_0 D\\ \big\{c(\nabla\varphi_0(\overline{a}) + \kappa_{\overline{a}}\nabla\varphi(\overline{a})) :\; c\ge 0\big\} & \textit{if } \varphi(a) < \varphi_0(\overline{a}) = 1\\ & \quad\textit{and } a\notin\partial_0 D\\ \big\{c\nabla\varphi(a) :\; c \ge 0\big\} & \textit{if either } a\in\partial_0 D\\ & \quad\textit{or } \varphi_0(\overline{a}) < \varphi(a) = 1\end{cases}$$

with

$$\kappa_a = -\left.\frac{\partial\varphi_0(\alpha,\beta)}{\partial\beta}\left(\frac{\partial\varphi(\alpha,\beta)}{\partial\beta}\right)^{-1}\right|_{(\alpha,\beta)=a}. \tag{2.4}$$

Proof. Indeed, under the hypotheses (H0)-(H4), the set

$$D\cap D_0 = \{a\in\mathbb{R}^d :\; \max\{\varphi(a),\varphi_0(a)\}\le 1\}$$

has a non-empty interior because $\varphi(0) = \varphi_0(0) = 1$ and

$$\frac{\nabla\varphi(0)}{|\nabla\varphi(0)|} + \frac{\nabla\varphi_0(0)}{|\nabla\varphi_0(0)|} \;=\; \frac{m}{|m|} + \frac{m_0}{|m_0|} \;\ne\; 0.$$

Since $D\cap D_0 \;\subset\; \hat{D} = (\Theta\times\mathbb{R})\cap D$, the set $\hat{D}$ has also a non-empty interior and by Corollary 23.8.1 of Rockafellar [19],

$$V(a) \;=\; V_D(a) + V_{\Theta\times\mathbb{R}}(a), \quad \forall a\in\hat{D} \tag{2.5}$$

where $V_{\Theta\times\mathbb{R}}(a) \subset \mathbb{R}^{d-1}\times\{0\}$ is the normal cone to the cylinder $\Theta\times\mathbb{R}$ at the point a and

$$V_D(a) \;=\; \begin{cases}\{c\nabla\varphi(a):\; c\geq 0\} & \text{if } \varphi(a)=1\\ \{0\} & \text{if } \varphi(a)<1\end{cases} \tag{2.6}$$

is a normal cone to the set D at the point a. Furthermore, recall that $\Theta\times\{0\}$ is the orthogonal projection of the set $D\cap D_0$ onto the hyper-plane $\mathbb{R}^{d-1}\times\{0\}$. since the orthogonal projection onto the hyper-plane $\mathbb{R}^{d-1}\times\{0\}$ of the point $a\in D$ is the same as the orthogonal projection of the point $\overline{a}$ from this it follows that

$$V_{\Theta\times\mathbb{R}}(a) \;=\; V_{\Theta\times\mathbb{R}}(\overline{a}) \;=\; V_{D\cap D_0}(\overline{a})\cap\left(\mathbb{R}^{d-1}\times\{0\}\right) \qquad \forall a\in\hat{D}$$

where $V_{D\cap D_0}(\overline{a})$ denotes the normal cone to the set $D\cap D_0$ at the point $\overline{a}\in\partial_-D$. Moreover, since $\varphi(\overline{a})=1$, using again Corollary 23.8.1 of Rockafellar [19] we get

$$V_{D\cap D_0}(\overline{a}) \;=\; V_D(\overline{a})+V_{D_0}(\overline{a}) \;=\; \begin{cases}\{c_1\nabla\varphi(\overline{a})+c_2\nabla\varphi_0(\overline{a}):\; c_i\geq 0\} & \text{if } \varphi_0(\overline{a})=1,\\ \{c\nabla\varphi(\overline{a}):\; c\geq 0\} & \text{if } \varphi_0(\overline{a})<1\end{cases}$$

and hence, for any $a\in\hat{D}$,

$$V_{\Theta\times\mathbb{R}}(a) \;=\; \begin{cases}\left\{c(\nabla\varphi_0(\overline{a})+\kappa_{\overline{a}}\nabla\varphi(\overline{a})):\; c\geq 0\right\} & \text{if } \varphi_0(\overline{a})=1 \text{ and } \overline{a}\notin\partial_0D,\\ \left\{c\nabla\varphi(\overline{a}):\; c\geq 0\right\} & \text{if } \overline{a}\in\partial_0D,\\ \{0\} & \text{if } \varphi_0(\overline{a})<1.\end{cases}$$

Finally, if $a\in\hat{D}$ and $\overline{a}\in\partial_0D$ then clearly $a=\overline{a}$, and consequently, the last relation combined with (2.5) and (2.6) prove (2.3). □

The next Lemma is needed to show that the mapping $q\to\hat{a}(q)$ is well defined.

Lemma 2.4. *Under the hypotheses (H0)-(H4), the set*

$$\Theta \doteq \{\alpha\in\mathbb{R}^{d-1}:\; \inf_{\beta\in\mathbb{R}}\max\{\varphi(\alpha,\beta),\varphi_0(\alpha,\beta)\}\leq 1\}$$

is strictly convex : for any two different points $\alpha,\alpha'\in\Theta$ and any $0<\theta<1$, the point $\alpha_\theta=\theta\alpha+(1-\theta)\alpha'$ belongs to the interior of the set Θ.

Proof. The set $\{a\in D:\varphi_0(a)\leq 1\}=\{a\in\mathbb{R}^d:\varphi(a)\leq 1 \text{ and } \varphi_0(a)\leq 1\}$ is compact and convex because the functions φ_0 and φ are continuous and convex on $\mathbb{R}^d$. The set Θ is therefore also compact and convex because $\Theta\times\{0\}$ is an orthogonal projection of the set $\{a\in D:\varphi_0(a)\leq 1\}$ on the hyper-plane $\mathbb{R}^{d-1}\times\{0\}$. Furthermore, remark that $\Theta\subset\{\alpha\in\mathbb{R}^{d-1}\;:\;\inf_\beta\varphi(\alpha,\beta)\leq 1\}$ and that the mapping $a=(\alpha,\beta)\to\alpha$ determines a homeomorphism between the set ∂_-D and the set $\{\alpha\in\mathbb{R}^{d-1},\,\inf_\beta\varphi(\alpha,\beta)\leq 1\}$. Let $\alpha\to(\alpha,\beta_\alpha)$ denote the inverse mapping to such a homeomorphism. Since for every $\alpha\in\mathbb{R}^{d-1}$, the function $\beta\to\varphi_0(\alpha,\beta)$ is increasing then a point $\alpha\in\mathbb{R}^{d-1}$ satisfying the inequality $\inf_\beta\varphi(\alpha,\beta)\leq 1$ belongs to the set Θ if and only if $\varphi_0(\alpha,\beta_\alpha)\leq 1$ and consequently,

$$\Theta \;=\; \{\alpha\in\mathbb{R}^{d-1}\;:\;\inf_\beta\varphi(\alpha,\beta)\leq 1 \text{ and } \varphi_0(\alpha,\beta_\alpha)\leq 1\}.$$

Under the hypotheses (H2), the set D is strictly convex because the function φ is strictly convex. The set $\{\alpha\in\mathbb{R}^{d-1}:\;\inf_\beta\varphi(\alpha,\beta)\leq 1\}$ is therefore also strictly convex and hence, to prove that the set Θ is strictly convex it is sufficient to show

that the function $\alpha \to \varphi_0(\alpha, \beta_\alpha)$ is strictly convex on $\{\alpha \in \mathbb{R}^{d-1} : \inf_\beta \varphi(\alpha,\beta) < 1\}$. For this we use Lemma 2.2. Recall that by Lemma 2.2, for $a = (\alpha, \beta_\alpha) \in \partial_- D$,

$$\varphi_0(\alpha, \beta_\alpha) \;=\; \mathbb{E}_0\big(\exp(a \cdot Z(\tau));\, \tau < \infty\big) \;=\; \sum_{x \in \mathbb{Z}^{d-1}} \mathbb{P}_0(X(\tau) = x)\; \exp(\alpha \cdot x)$$

where τ is the first time when the process $Z(t) = (X(t), Y(t))$ returns to the boundary hyper-plane $\mathbb{Z}^{d-1} \times \{0\}$. Since under the hypotheses of our lemma, the function φ_0 is finite everywhere on $\mathbb{R}^d$ then the series at the right hand side of the above relation converge on $\{\alpha \in \mathbb{R}^{d-1} \;:\; \inf_\beta \varphi(\alpha,\beta) \le 1\}$. By dominated convergence theorem, from this it follows that the function $\alpha \to \varphi_0(\alpha, \beta_\alpha)$ is infinitely differentiable on $\{\alpha \in \mathbb{R}^{d-1} \;:\; \inf_\beta \varphi(\alpha,\beta) < 1\}$ and that its Hessian matrix

$$Q(\alpha) \;=\; \left(\frac{\partial^2 \varphi_0(\alpha, \beta_\alpha)}{\partial \alpha_i \partial \alpha_j}\right)_{1 \le i,j \le d-1}$$

satisfies the equality

$$\xi \cdot Q(\alpha)\xi \;=\; \sum_{x \in \mathbb{Z}^{d-1}} e^{\alpha \cdot x}\; (\xi \cdot x)^2 \mathbb{P}_0(X(\tau) = x)$$

for any $\xi \in \mathbb{R}^{d-1}$ whenever $\inf_\beta \varphi(\alpha,\beta) < 1$. Since the Markov process $Z(t)$ is irreducible, then for every non-zero vector $\xi \in \mathbb{R}^{d-1}$ there is $x \in \mathbb{Z}^{d-1}$ such that $(\xi \cdot x)^2 \mathbb{P}_0(X(\tau) = x) \;>\; 0$ and consequently, $\xi \cdot Q(\alpha)\xi > 0$. This proves that the function $\alpha \to \varphi(\alpha, \beta_\alpha)$ is strictly convex on the set $\{\alpha \in \mathbb{R}^{d-1} :\; \inf_\beta \varphi(\alpha,\beta) < 1\}$. Lemma 2.4 is proved. $\square$

We are ready now to get the following statement.

Lemma 2.5. *Under the hypotheses (H0)-(H4), for every non-zero vector $q \in \mathbb{R}^{d-1} \times \mathbb{R}_+$, there is a unique point $\hat{a}(q) \in \hat{D} \cap \partial_+ D$ for which $q \in V(\hat{a}(q))$.*

Proof. Recall that for $\hat{a} \in \hat{D}$, the vector q belongs to the normal cone $V(\hat{a})$ to the set $\hat{D}$ if and only if

$$\sup_{a \in \hat{D}} \; a \cdot q \;=\; \hat{a} \cdot q. \tag{2.7}$$

Since under the hypotheses (H0)-(H4), the set $\hat{D}$ is compact and non-empty, for every $q \in \mathcal{S}^d$ there is $\hat{a} = \hat{a}(q) \in \hat{D}$ for which this equality holds. It is clear that for $q \neq 0$, such a point $\hat{a}(q)$ belongs to the boundary $\partial\hat{D}$ of the set $\hat{D}$. Moreover, Lemma 2.3 shows that for $q \in \mathbb{R}^{d-1} \times]0, +\infty[$,

$$\hat{a}(q) \in \hat{D} \cap \partial_+ D.$$

For $q \in \mathbb{R}^{d-1} \times \{0\}$, a point $\hat{a} = \hat{a}(q)$ satisfying the equality (2.7) can be non-unique : if the equality (2.7) holds for some $\hat{a} \in \partial(\hat{D})$ then

$$\sup_{a \in \hat{D}} \; a \cdot q \;=\; \tilde{a} \cdot q$$

for all $\tilde{a} \in \partial\hat{D}$ having the same first $d-1$ coordinates as the point $\hat{a}$. Remark however that for every $a \in \partial\hat{D}$, there is a unique point $\hat{a} \in \hat{D} \cap \partial_+ D$ with the same first $d-1$ coordinates as the point a and hence without any restriction of generality we can assume that $\hat{a}(q) \in \hat{D} \cap \partial_+ D$.

We have shown that for every non-zero vector $q \in \mathbb{R}^{d-1} \times \mathbb{R}_+$, there is a point $\hat{a}(q) \in \hat{D} \cap \partial_+ D$ for which $q \in V(\hat{a}(q))$. To complete the proof of our lemma it

is now sufficient to show that such a point is unique. Suppose that there are two different points $\hat a(q), \tilde a(q) \in \hat D \cap \partial_+ D$ for which $q \in V(\hat a(q)) \cap V(\tilde a(q))$. Then

$$\sup_{a\in\hat D} a\cdot q \;=\; \tilde a(q)\cdot q \;=\; \hat a(q)\cdot q$$

and consequently, for every $\theta\in[0,1]$,

$$\sup_{a\in\hat D} a\cdot q \;=\; \big(\theta\tilde a(q) + (1-\theta)\hat a(q)\big)\cdot q.$$

The last equality shows that the point $a_\theta \doteq \theta\tilde a(q) + (1-\theta)\hat a(q)$ belongs to the boundary of the set $\hat D$ and that $q\in V(a_\theta)$. Recall now that under the hypotheses of our lemma, the set D is strictly convex and consequently, for $0<\theta<1$, the point $a_\theta \doteq \theta\tilde a(q) + (1-\theta)\hat a(q)$ belongs to the interior of the set D. Hence, the normal cone $V_D(a_\theta)$ to the set D at the point a_θ is zero and the normal cone $V(a_\theta)$ to $\hat D$ at the point a_θ coincide with the normal cone $V_{\Theta\times\mathbb{R}}(a_\theta)$ to the set $\Theta\times\mathbb{R}$ at a_θ (this is a consequence of Corollary 23.8.1 of [19]). From this it follows that

$$q\in V(a_\theta) \;=\; V_{\Theta\times\mathbb{R}}(a_\theta) \subset \mathbb{R}^{d-1}\times\{0\}. \tag{2.8}$$

For $q\in\mathbb{R}^{d-1}\times]0,+\infty[$, the point $\hat a(q) = \tilde a(q)$ is therefore unique. For $q\in\mathbb{R}^{d-1}\times\{0\}$, (2.8) shows that the first $d-1$ coordinates of the points a_θ and $a_{\theta'}$ are the same for all $0<\theta<\theta'<1$ because by Lemma 2.4, the set Θ is strictly convex. Letting $\theta\to 0$ and $\theta'\to 1$ we conclude that the first $d-1$ coordinates of the points $\hat a(q))$ and $\tilde a(q)$ are the same. This proves that $\hat a(q) = \tilde a(q)$ because $\hat a(q), \tilde a(q)\in\partial_+ D$ and the orthogonal projection determines a one to one mapping from $\partial_+ D$ to $\mathbb{R}^{d-1}\times\{0\}$. □

Lemma 2.3 and Lemma 2.5 imply the following statement.

Corollary 2.4. *Under the hypotheses (H0)-(H4), for every $q\in\mathbb{R}^{d-1}\times]0,+\infty[$, the following assertions hold :*

1) *there is a unique vector $\gamma_q\in\mathbb{R}^{d-1}\times\{0\}$ for which the vector $q-\gamma_q$ belongs to the normal cone to the set D at the point $\hat a(q)$ and γ_q, $q-\gamma_q\in V(\hat a(q))$.*
2) $\varphi_0\left(\overline{\hat a(q)}\right) = 1$ *whenever $\gamma_q\neq 0$.*

3. Harmonic functions

The harmonic function of the Markov process $(Z(t))$ are now identified. The main result of this section is the following proposition.

Proposition 3.1. *Under the hypotheses (H0)-(H4), the following assertions hold. 1) A non-negative function h is harmonic for the Markov process $(Z(t))$ if and only if there is a positive measure ν_h on $\hat D\cap\partial_+ D = (\Theta\times\mathbb{R})\cap\partial_+ D$ such that*

$$h(z) \;=\; \int_{(\Theta\times\mathbb{R})\cap\partial_+ D} h_a(z)\,d\nu_h(a), \qquad \forall z\in\mathbb{N}^*\times\mathbb{Z}^{d-1}. \tag{3.1}$$

2) For every $a=(\alpha,\beta)\in(\Theta\times\mathbb{R})\cap\partial_+ D$ with $\alpha\in\mathbb{R}^{d-1}$ and $\beta\in\mathbb{R}$, the constant multiples of the function h_a defined by (1.8) are the only non-negative harmonic functions for which

$$\sup_{x\in\mathbb{R}^{d-1}} \exp(-\alpha\cdot x)h(x,y) \;<\; +\infty, \qquad \forall y\in\mathbb{N}. \tag{3.2}$$

3) The constant multiples of the functions h_a with $a\in(\Theta\times\mathbb{R})\cap\partial_+ D$, are the only minimal harmonic functions of the Markov process $(Z(t))$.

In order to prove this result we use the properties of Markov-additive processes. Recall that a Markov process $(A(t), M(t))$ on a countable set $\mathbb{Z}^d \times E$ with transition probabilities $p\big((x,y),(x',y')\big)$ is called *Markov-additive* if

$$p\big((x,y),(x',y')\big) \;=\; p\big((0,y),(x'-x,y')\big)$$

for all $x, x' \in \mathbb{Z}^d$, $y, y' \in E$. The first component $A(t)$ is an *additive* part of the process $(A(t), M(t))$, and $M(t)$ is its *Markovian part*.

According to this definition, the Markov process $Z(t) = (X(t), Y(t))$ is Markov-additive with an additive part $X(t)$ taking the values in $\mathbb{Z}^{d-1}$ and Markovian part $Y(t)$ taking the values in $\mathbb{N}$. Under the hypotheses (H1), its *Feynman-Kac transform* matrix $\mathcal{P}(\alpha) = \big(\mathcal{P}(\alpha, y, y'),\ y, y' \in \mathbb{N}\big)$ with $\alpha \in \mathbb{R}^{d-1}$ and

$$\mathcal{P}(\alpha, y, y') \;=\; \mathbb{E}_{(0,y)}\big(\exp(\alpha \cdot X(1));\, Y(1) = y'\big)$$

is irreducible and the limit

$$\lambda(\alpha) \;=\; \limsup_n \frac{1}{n} \log \mathcal{P}^{(n)}(\alpha, y, y')$$

does not depend on $y, y' \in \mathbb{N}$ (see [21]). The quantity $e^{\lambda(\alpha)}$ is usually called *spectral radius* and $e^{-\lambda(\alpha)}$ is the *convergence parameter* of the transform matrix $\mathcal{P}(\alpha)$. By Proposition 3.1 of Ignatiouk [15], every non-zero minimal harmonic function h of the Markov process $Z(t)$ is of the form

$$h(x,y) = \exp(\alpha \cdot x) h(0,y), \qquad \forall (x,y) \in \mathbb{Z}^{d-1} \times \mathbb{N}. \tag{3.3}$$

with some $\alpha \in \mathbb{R}^{d-1}$ satisfying the inequality $\lambda(\alpha) \le 0$. The following lemma identifies the function $\alpha \to \lambda(\alpha)$.

Lemma 3.1. *Under the hypotheses* $(H0) - (H4)$,

$$\lambda(\alpha) \;=\; \inf_{\beta \in \mathbb{R}} \log \max\{\varphi(\alpha,\beta), \varphi_0(\alpha,\beta)\}, \qquad \forall \alpha \in \mathbb{R}^{d-1}. \tag{3.4}$$

Proof. Remark first of all that for any $(\alpha,\beta) \in \mathbb{R}^{d-1} \times \mathbb{R}$, the exponential function $f(y) = \exp(\beta y)$ on $\mathbb{N}$ satisfies the inequality

$$\mathcal{P}(\alpha) f(y) \;=\; \mathbb{E}_{(0,y)}(\exp(\alpha \cdot X(1) + \beta Y(1))) \le \max\{\varphi(\alpha,\beta), \varphi_0(\alpha,\beta)\} f(y) \quad \forall y \in \mathbb{N}.$$

From this it follows that $\lambda(\alpha) \le \log \max\{\varphi(\alpha,\beta), \varphi_0(\alpha,\beta)\}$ for all $(\alpha,\beta) \in \mathbb{R}^{d-1} \times \mathbb{R}$ (see Seneta [21] for more details) and consequently,

$$\lambda(\alpha) \;\le\; \inf_{\beta \in \mathbb{R}} \log \max\{\varphi(\alpha,\beta), \varphi_0(\alpha,\beta)\}, \qquad \forall \alpha \in \mathbb{R}^{d-1}.$$

Furthermore, let τ denote the first time when the process $(Z(t))$ hits the boundary hyperplane $Z^{d-1} \times \{0\}$. Then for $y, y' > 0$, $y, y' \in \mathbb{N}$,

$$\begin{aligned}
\lambda(\alpha) \;&=\; \limsup_n \frac{1}{n} \log \mathbb{E}_{(0,y)}\big(\exp(\alpha \cdot X(n));\, Y(n) = y'\big) \\
&\ge\; \limsup_n \frac{1}{n} \log \mathbb{E}_{(0,y)}\big(\exp(\alpha \cdot X(n));\, Y(n) = y',\ \tau > n\big) \;=\; \inf_{\beta \in \mathbb{R}} \log \varphi(\alpha,\beta)
\end{aligned}$$

where the last relation is proved by Lemma 5.1 of Ignatiouk [15]. For those $\alpha \in \mathbb{R}^{d-1}$ for which the right hand side of (3.4) is equal to right hand side of the last relation, the equality (3.4) is therefore verified. Suppose now that

$$\inf_{\beta \in \mathbb{R}} \max\{\varphi(\alpha,\beta), \varphi_0(\alpha,\beta)\} \;>\; \inf_{\beta \in \mathbb{R}} \varphi(\alpha,\beta).$$

In this case, the minimum of the function $\beta \to \max\{\varphi(\alpha,\beta), \varphi_0(\alpha,\beta)\}$ is achieved at a point $\hat\beta_\alpha \in \mathbb{R}$ where

$$\varphi(\alpha,\hat\beta_\alpha) = \varphi_0(\alpha,\hat\beta_\alpha) \qquad \text{and} \qquad \frac{\partial}{\partial\beta}\varphi(\alpha,\hat\beta_\alpha) < 0.$$

Under the hypotheses $(H0)-(H4)$, the twisted Markov chain $(\tilde Y(t))$ on $\mathbb{N}$ having transition probabilities $\tilde p(y,y') = \mathcal{P}(\alpha,y,y')\exp(\hat\beta_\alpha(y'-y))/\varphi(\alpha,\hat\beta_\alpha)$ is irreducible and satisfies the conditions of Foster's criterion of positive recurrence (see Corollary 8.7 in [20]) with the test function $f(y) = y$:

$$\mathbb{E}_0(\tilde Y(1)) \ < \ +\infty \quad \text{and} \quad \mathbb{E}_y(\tilde Y(1)) \ = \ y + \frac{\partial}{\partial\beta}\varphi(\alpha,\hat\beta_\alpha) \ < \ y, \quad \forall y > 0.$$

The Markov chain $(\tilde Y(t))$ is therefore positive recurrent and consequently,

$$\limsup_{n\to\infty} \frac{1}{n}\log\tilde p^{(n)}(y,y') \ = \ 0, \qquad \forall y,y' \in \mathbb{N}.$$

The last relation together with the equality $\mathcal{P}^{(n)}(\alpha,y,y) = \tilde p^{(n)}(y,y)(\varphi(\alpha,\hat\beta_\alpha))^n$ shows that $\lambda(\alpha) = \log\varphi(\alpha,\hat\beta_\alpha)$ from which it follows (3.4). □

Lemma 3.1 proves that $\lambda(\alpha) \le 0$ if and only if $\alpha \in \Theta$ and hence, using Proposition 3.1 of Ignatiouk [15] we get

Corollary 3.1. *Under the hypotheses $(H0)-(H4)$, every minimal harmonic function h of the Markov process $(Z(t))$ satisfies the equality* (3.3) *with some $\alpha \in \Theta$.*

Now we identify the minimal harmonic functions satisfying the equality (3.3).

Lemma 3.2. *Under the hypotheses (H0)-(H4), for every point $a = (\alpha,\beta) \in (\Theta \times \mathbb{R})\cap\partial_+ D$, the constant multiples of h_a are the only minimal non-negative harmonic functions of the Markov process $(Z(t))$ for which the equality* (3.3) *holds with a given $\alpha \in \Theta$.*

Proof. Let $a = (\alpha,\beta) \in (\Theta\times\mathbb{R})\cap\partial_+ D$. Straightforward calculation shows that the function h_a is non-negative and harmonic for the Markov process $(Z(t))$. Recall that a non-zero harmonic function $h \ge 0$ is called *minimal* if for any non-zero harmonic function $h' \ge 0$, the inequality $h' \le h$ implies that $h' = ch$ with some constant $c > 0$. To prove our Lemma it is therefore sufficient to show that if $h \ne 0$ is a minimal non-negative harmonic functions of the Markov process $(Z(t))$ for which (3.3) holds with a given α then

$$h \ge ch_a \tag{3.5}$$

with some $c > 0$. For this we first show that every such a function $h \ne 0$ satisfies the inequality

$$h(z) \ \ge \ h(0)\exp(\overline{a}\cdot z) \ > \ 0 \qquad \text{for all } z \in \mathbb{Z}^{d-1}\times\mathbb{N}. \tag{3.6}$$

Indeed, let h be a non-zero minimal non-negative harmonic functions for which the equality (3.3) holds with a given α. Then $h(z) > 0$ for all $z \in \mathbb{Z}^{d-1}\times\mathbb{N}$ because the Markov process $Z(t)$ is irreducible. Moreover, according to the definition of the mapping $a \to \overline{a}$, from (3.3) it follows that

$$h(z) \ = \ h(0)\exp(\alpha\cdot x) \ = \ h(0)\exp(\overline{a}\cdot z) \ > \ 0 \quad \text{for any } z = (x,0) \in \mathbb{Z}^{d-1}\times\{0\}. \tag{3.7}$$

Hence, for $z \in \mathbb{Z}^{d-1} \times \{0\}$ the inequality (3.6) holds with the equality. Furthermore, for $\tau = \inf\{t > 0 : Y(t) = 0\}$, the sequence $h(Z(n \wedge \tau))$ is a martingale relative to the natural filtration and $h(Z(n\wedge\tau)) = h(0)\exp(\alpha \cdot X(\tau))$ whenever $\tau \le n$. Hence, for any $z = (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}$ with $y > 0$ we have

$$h(z) \;=\; \mathbb{E}_z(h(Z(n\wedge\tau)) \;\ge\; h(0)E_z\Big(\exp(\alpha\cdot X(\tau));\; \tau \le n\Big), \quad \forall n \in \mathbb{N}$$

and consequently, letting $n \to \infty$ and using Fatou lemma we obtain

$$h(x,y) \;\ge\; h(0)E_z\Big(\exp(\alpha\cdot X(\tau));\; \tau < \infty\Big)$$

The last inequality combined with Lemma 2.2 proves (3.6) for $z = (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}$ with $y > 0$. The inequality (3.6) is therefore verified.

Recall now that $(\Theta \times \mathbb{R}) \cap \partial_+ D \;=\; \{a \in \partial_+ D : \;\varphi_0(\overline{a}) \le 1\}$ where $\overline{a}$ is a point on the boundary $\partial_- D$ having the same $d-1$ first coordinates as the point a. From now on the proof of (3.5) is different in each of the following cases :

- *case 1* : when $\varphi_0(\overline{a}) = 1$,
- *case 2* : when $\varphi_0(\overline{a}) < 1$.

If $\varphi_0(\overline{a}) = 1$ then from (1.8) it follows that $h_a(z) = \exp(\overline{a}\cdot z)$ for all $z \in \mathbb{Z}^{d-1} \times \mathbb{N}$ and hence, the inequality (3.6) proves (3.5) with $c = h(0)$. For all those $a = (\alpha, \beta) \in (\Theta \times \mathbb{R}) \cap \partial_+ D$ for which $\varphi_0(\overline{a}) = 1$, Lemma 3.2 is therefore proved.

Suppose now that $\varphi_0(\overline{a}) < 1$ and let $h_+(z) = h(z) - h(0)\exp(\overline{a}\cdot z)$. Then the inequality (3.6) shows that the function h_+ is non-negative, the equality (3.7) implies that

$$h_+(z) \;=\; 0, \qquad \text{for any } z = (x, 0) \in \mathbb{Z}^{d-1} \times \{0\}, \tag{3.8}$$

and from the equality (3.3) it follows that

$$h_+(x,y) = \exp(\alpha\cdot x) h_+(0,y), \qquad \text{for all } z = (x,y) \in \mathbb{Z}^{d-1} \times \mathbb{N}. \tag{3.9}$$

Moreover, straightforward calculations show that for $z = (x, 0) \in \mathbb{Z}^{d-1} \times \{0\}$,

$$\mathbb{E}_z(h_+(Z(1))) \;=\; (1 - \varphi_0(\overline{a}))\exp(a\cdot z)h(0) \;>\; 0, \quad \forall x \in \mathbb{Z}^{d-1}, \tag{3.10}$$

and for $z = (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}$ with $y > 0$,

$$\mathbb{E}_z(h_+(Z(1))) \;=\; h(z) - \varphi(\overline{a})\exp(\overline{a}\cdot z)h(0) \;=\; h(z) - \exp(\overline{a}\cdot z)h(0) \;=\; h_+(z). \tag{3.11}$$

According to the definition of the Markov process $(Z(t))$, relations (3.8) and (3.11) show that the function h_+ satisfies the equality

$$\sum_{z'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*} \mu(z'-z)h_+(z') \;=\; h_+(z) \qquad \forall z \in \mathbb{Z}^{d-1}\times\mathbb{N}^*, \tag{3.12}$$

and from (3.10) it follows that $h_+ \not\equiv 0$. Under the hypotheses of our lemma, Proposition 2.1 and Proposition 5.1 of Ignatiouk [15] prove that the only non-negative non-zero functions satisfying the equalities (3.9) and (3.12) are the constant multiples of

$$h_{a,+}(z) \;=\; \begin{cases} \exp(a\cdot z) - \exp(\overline{a}\cdot z) & \text{if } a \notin \partial_0 D, \\ y\;\exp(a\cdot z) & \text{if } a \in \partial_0 D, \qquad z = (x,y) \in \mathbb{Z}^d \times \mathbb{N}^*. \end{cases}$$

Hence, $h_+(x, y) = ch_{a,+}(x, y)$ for all $(x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}^*$ with some $c > 0$, and consequently,

$$h(z) = h(0)\exp(\overline{a}\cdot z) + ch_{a,+}(z) \qquad \forall\, z \in \mathbb{Z}^{d-1}\times\mathbb{N}^*.$$

To complete the proof of our lemma it is sufficient now to notice that $h_a(z) = C\exp(\overline{a}\cdot z) + h_{a;+}(z)$ with

$$C \;=\; \begin{cases} \frac{\partial}{\partial\beta}\varphi_0(a)/(1-\varphi_0(\overline{a})) & \text{if } a\in\partial_0 D \\ (\varphi_0(a)-\varphi_0(\overline{a}))/(1-\varphi_0(\overline{a}) & \text{otherwise} \end{cases}$$

from which it follows that $h(z) \geq \min\{c, h(0)/C\}h_a(z)$. □

Lemma 3.2 combined with Corollary 3.1 implies the following statement.

Corollary 3.2. *Under the hypotheses (H0)-(H4), every minimal harmonic function of the Markov process $(Z(t))$ is of the form $h = ch_a$ with some $c > 0$ and $a \in (\Theta\times\mathbb{R})\cap\partial_+ D$.*

Proof. To get this statement from Corollary 3.1 and Lemma 3.2 it is sufficient to notice that the orthogonal projection onto the hyper-plane $\mathbb{R}^{d-1}\times\{0\}$ determines a homeomorphism from $(\Theta\times\mathbb{R})\cap\partial_+ D$ to $\Theta\times\{0\}$. □

Proof of Proposition 3.1. The proof of this proposition uses Corollary 3.2 and the same arguments as in the proof of Proposition 5.1 of Ignatiouk [15]. The main steps of this proof are the following.

By the Poisson-Martin representation theorem (see Woess [23]), every non-negative harmonic function of the Markov process $(Z(t))$ is of the form

$$h(z) \;=\; \int_{\partial_m(\mathbb{Z}^{d-1}\times\mathbb{N})} K(z,\gamma)\, d\tilde{\nu}_h(\gamma), \qquad \forall z\in\mathbb{Z}^{d-1}\times\mathbb{N}^*$$

with some Borel measure $\tilde{\nu}_h \geq 0$ on the minimal Martin boundary $\partial_m(\mathbb{Z}^{d-1}\times\mathbb{N})$. Recall that $K(z,\gamma)$ is the Martin kernel of the Markov process $(Z(t))$, the mapping $\gamma\to K(z,\gamma)$ is continuous on $\partial_m(\mathbb{Z}^{d-1}\times\mathbb{N})$ for every $z\in\mathbb{Z}^{d-1}\times\mathbb{N}$ and for every $\gamma\in\partial_m(\mathbb{Z}^{d-1}\times\mathbb{N})$, according to the definition of the minimal Martin boundary (see Woess [23]), the function $z\to K(z,\gamma)$ is a minimal harmonic function for the Markov process $(Z(t))$ with $K(z_0,\gamma) = 1$. By Corollary 3.2, we have therefore

$$K(z,\gamma) = c_\gamma h_{a(\gamma)}(z) \quad \text{for all } z\in\mathbb{Z}^{d-1}\times\mathbb{N}$$

with some $a(\gamma) = (\alpha(\gamma),\beta(\gamma))\in(\Theta\times\mathbb{R})\cap\partial_+ D$ and $c_\gamma = 1/h_{a(\gamma)}(z_0)$. For $z_0 = (x_0,y_0)$ and $z = (x,y_0)$, the mapping

$$\gamma \;\to\; K(z,\gamma) \;=\; \exp\bigl(\alpha(\gamma)\cdot(x-x_0)\bigr) \;>\; 0$$

is therefore continuous on $\partial_m(\mathbb{Z}^{d-1}\times\mathbb{N}^*)$ for any $x\in\mathbb{Z}^{d-1}$. This proves that the mapping $\gamma\to\alpha(\gamma)$ from $\partial_m(\mathbb{Z}^{d-1}\times\mathbb{N}^*)$ to Θ is continuous. The mapping $\gamma\to a(\gamma)\in(\Theta\times\mathbb{R})\cap\partial_+ D$ is therefore also continuous on $\partial_m(\mathbb{Z}^{d-1}\times\mathbb{N}^*)$ because the mapping $(\alpha,\beta)\to\alpha$ defines a homeomorphism from $(\Theta\times\mathbb{R})\cap\partial_+ D$ to Θ. From this it follows that the integral representation (3.1) holds with the positive Borel measure ν_h on $(\Theta\times\mathbb{R})\cap\partial_+ D$ defined by

$$\nu_h(B) \;=\; \int_{\{\gamma: a(\gamma)\in B\}} c_\gamma\, d\tilde{\nu}_h(\gamma)$$

for every Borel subset $B\subset(\Theta\times\mathbb{R})\cap\partial_+ D$. The first assertion of Proposition 3.1 is therefore proved.

To prove the second assertion it is sufficient to show that a non-zero harmonic function $h\geq 0$ satisfies (3.2) with some $a = (\alpha,\beta)\in(\Theta\times\mathbb{R})\cap\partial_+ D$ if and only if

$$\mathrm{supp}(\nu_h) = \{a\}.$$

For every $a \in (\Theta \times \mathbb{R}) \cap \partial_+ D$, the function h_a satisfies (3.2) and is harmonic for the Markov process $(Z(t))$. Conversely, if $\mathrm{supp}(\nu_h) \neq \{\hat a\}$ for some $\hat a = (\hat\alpha, \hat\beta) \in (\Theta \times \mathbb{R}) \cap \partial_+ D$, then there is an open ball $B(a_0, \varepsilon)$ in $\mathbb{R}^d$ centered at some point $a_0 = (\alpha_0, \beta_0) \in (\Theta \times \mathbb{R}) \cap \partial_+ D$ and having a radius $\varepsilon > 0$ such that $\nu_h(B(a_0,\varepsilon) \cap (\Theta \times \mathbb{R}) \cap \partial_+ D) > 0$ and there is $x_0 \in \mathbb{Z}^{d-1}$ such that $(\alpha - \hat\alpha)\cdot x_0 > 0$ for all $a = (\alpha, \beta) \in B(a_0, \varepsilon)$. Using the integral representation (3.1) and Fatou lemma from this it follows that

$$\begin{aligned}
\sup_{x \in \mathbb{R}^{d-1}} e^{-\hat\alpha\cdot x} h(x,y) &\geq \limsup_{n\to\infty} e^{-n\hat\alpha\cdot x_0} h(nx_0, y) \\
&\geq \limsup_{n\to\infty} \int_{B(a_0,\varepsilon)\cap(\Theta\times\mathbb{R})\cap\partial_+ D} e^{n(\alpha-\hat\alpha)\cdot x_0} h_{a,+}(0,y)\, d\nu_h(a) \\
&\geq \int_{B(a_0,\varepsilon)\cap(\Theta\times\mathbb{R})\cap\partial_+ D} \lim_{n\to\infty} e^{n(\alpha-\hat\alpha)\cdot x_0} h_{a,+}(0,y)\, d\nu_h(a) = +\infty.
\end{aligned}$$

The second assertion of Proposition 3.1 is proved.

Finally, if a non-negative harmonic function h satisfies the inequality $h \leq h_a$ for some $a \in (\Theta \times \mathbb{R}) \cap \partial_+ D$ then for h the inequality (3.2) holds with the same a and consequently $h = ch_a$ for some $c \geq 0$. For every $a \in (\Theta \times \mathbb{R}) \cap \partial_+ D$, the harmonic function $h_a > 0$ is therefore minimal and conversely, by Corollary 3.2, every minimal harmonic function of the Markov process $(Z(t))$ is of the form ch_a with some $c > 0$ and $a \in (\Theta \times \mathbb{R}) \cap \partial_+ D$. Proposition 3.1 is proved.

4. Communication condition

Definition : *A discrete time Markov chain $(Z(t))$ on $\mathbb{Z}^d$ is said to satisfy communication condition on $E \subset \mathbb{Z}^d$ if there exist $\theta > 0$ and $C > 0$ such that for any $z, z' \in E$ there is a sequence of points $z_0, z_1, \ldots, z_n \in E$ with $z_0 = z$, $z_n = z'$ and $n \leq C|z' - z|$ such that $|z_i - z_{i-1}| \leq C$ and $\mathbb{P}_{z_{i-1}}(Z(1) = z_i) \geq \theta$ for all $i = 1, \ldots, n$.*

Proposition 4.1. *Under the hypotheses (H0)-(H3), the Markov process $(Z(t))$ satisfies communication condition on $\mathbb{Z}^{d-1} \times \mathbb{N}$.*

Proof. Recall that on the half-space $\mathbb{Z}^{d-1} \times \mathbb{N}^*$, the Markov process $(Z(t))$ behaves as a homogeneous random walk $(S(t))$ on $\mathbb{Z}^d$ having transition probabilities $p(z,z') = \mu(z'-z)$. Let $(Z_+(t))$ denote a sub-stochastic random walk on $\mathbb{Z}^{d-1} \times \mathbb{N}^*$ with transition matrix $\big(p(z,z') = \mu(z'-z),\ z, z' \in \mathbb{Z}^{d-1} \times \mathbb{N}^*\big)$. Such a Markov process is identical to the homogeneous random walk $(S(t))$ until the first time when $(S(t))$ hits the boundary hyperplane $\mathbb{Z}^{d-1} \times \{0\}$ and dies when $(S(t))$ hits $\mathbb{Z}^{d-1} \times \{0\}$. By Lemma 4.1 of Ignatiouk [15], the Markov process $(Z_+(t))$ satisfies communication condition on $\mathbb{Z}^{d-1} \times \mathbb{N}^*$: there exist $\theta > 0$ and $C > 0$ such that for any $z, z' \in \mathbb{Z}^{d-1} \times \mathbb{N}^*$ there is a sequence of points $z_0, z_1, \ldots, z_n \in \mathbb{Z}^{d-1} \times \mathbb{N}^*$ with $z_0 = z$, $z_n = z'$ and $n \leq C|z' - z|$ such that

$$|z_i - z_{i-1}| \leq C \quad \text{and} \quad \mu(z_i - z_{i-1}) \geq \theta, \quad \forall\, i = 1, \ldots, n.$$

Since the Markov process $(Z(t))$ has the same transition probabilities on the set $\mathbb{Z}^{d-1} \times \mathbb{N}^*$ as $(Z_+(t))$, we conclude that $(Z(t))$ also satisfies communication condition on $\mathbb{Z}^{d-1} \times \mathbb{N}^*$ with the same constants $C > 0$ and $\theta > 0$. Moreover, the Markov process $(Z(t))$ is irreducible and its transition probabilities are invariant with respect to the shifts on $z \in \mathbb{Z}^{d-1} \times \{0\}$. Hence, there are $w, w' \in \mathbb{Z}^{d-1} \times \mathbb{N}^*$

such that $p(z,z+w) = \mu_0(w) > 0$ and $p(z+w',z) = \mu(-w') > 0$ for all $z \in \mathbb{Z}^{d-1}\times\{0\}$. From this it follows that the Markov process $(Z(t))$ satisfies communication condition on $\mathbb{Z}^{d-1}\times\mathbb{N}$ with another constants $C' = C + |w| + |w'|$ and $\theta' = \min\{\theta, \mu_0(w), \mu(-w')\}$: for any $z, z' \in \mathbb{Z}^{d-1}\times\mathbb{N}$ there is a sequence of points $z_0, z_1, \ldots, z_n \in \mathbb{Z}^{d-1}\times\mathbb{N}$ with $z_0 = z$, $z_n = z'$ and $n \le C'|z'-z|$ such that $|z_i - z_{i-1}| \le C'$ and $\mathbb{P}_{z_{i-1}}(Z(1) = z_i) \ \ge\ \theta'$ for all $i = 1,\ldots,n$ where $z_1 = z+w$ if $z \in \mathbb{Z}^{d-1}\times\{0\}$ and $z_{n-1} = z' - w'$ if $z' \in \mathbb{Z}^{d-1}\times\{0\}$. □

5. Large deviation estimates

In this section, we obtain large deviation estimates for Green's function of the Markov processes $(Z(t))$ and $(Z_+(t))$ by using sample path large deviation properties of scaled processes $Z^\varepsilon(t) \ = \ \varepsilon Z([t/\varepsilon])$ and $Z^\varepsilon_+(t) = \varepsilon Z_+([t/\varepsilon])$. Recall that $(Z_+(t))$ is a sub-stochastic random walk on the half-space $\mathbb{Z}^{d-1}\times\mathbb{N}^*$ having transition matrix

$$\left(p(z,z') = \mu(z'-z),\ z,z' \in \mathbb{Z}^{d-1}\times\mathbb{N}^*\right).$$

The random walk $(Z_+(t))$ is identical to the homogeneous random walk on $\mathbb{Z}^d$ killed upon hitting the boundary hyper-plane $\mathbb{Z}^{d-1}\times\{0\}$.

5.1. Sample path large deviation principle for scaled processes.

Before to formulate our large deviation results we recall the definition of the sample path large deviation principle.

Definitions : *1) Let $D([0,T],\mathbb{R}^d)$ denote the set of all right continuous with left limits functions from $[0,T]$ to $\mathbb{R}^d$ endowed with Skorohod metric (see Billingsley [1]). Recall that a mapping $I_{[0,T]} :\ D([0,T],\mathbb{R}^d) \to [0,+\infty]$ is a good rate function on $D([0,T],\mathbb{R}^d)$ if for any $c \ge 0$ and any compact set $V \subset \mathbb{R}^d$, the set*

$$\{\varphi \in D([0,T],\mathbb{R}^d) :\ \phi(0) \in V \ \ \textit{and}\ \ I_{[0,T]}(\varphi) \le c\}$$

is compact in $D([0,T],\mathbb{R}^d)$. According to this definition, a good rate function is lower semi-continuous.

2) For a Markov chain $(Z(t))$ on $E \subset \mathbb{R}^d$ the family of scaled processes $(Z^\varepsilon(t) = \varepsilon Z([t/\varepsilon]),\ t \in [0,T])$, is said to satisfy sample path large deviation principle in $D([0,T],\mathbb{R}^d)$ with a rate function $I_{[0,T]}$ if for any $z \in \mathbb{R}^d$

$$\lim_{\delta\to 0}\ \liminf_{\varepsilon\to 0}\ \inf_{z'\in E:|\varepsilon z'-z|<\delta} \varepsilon\log\mathbb{P}_{z'}\left(Z^\varepsilon(\cdot)\in\mathcal{O}\right) \ge -\inf_{\phi\in\mathcal{O}:\phi(0)=z} I_{[0,T]}(\phi), \tag{5.1}$$

for every open set $\mathcal{O} \subset D([0,T],\mathbb{R}^d)$, and

$$\lim_{\delta\to 0}\ \limsup_{\varepsilon\to 0}\ \sup_{z'\in E:|\varepsilon z'-z|<\delta} \varepsilon\log\mathbb{P}_{z'}\left(Z^\varepsilon(\cdot)\in F\right) \le -\inf_{\phi\in F:\phi(0)=z} I_{[0,T]}(\phi). \tag{5.2}$$

for every closed set $F \subset D([0,T],\mathbb{R}^d)$.

We refer to sample path large deviation principle as SPLD principle. Inequalities (5.1) and (5.2) are referred as lower and upper SPLD bounds respectively.

Proposition 4.1 of Ignatiouk [15] proves that under the hypotheses (H2) and (H4), the family of scaled processes $(Z^\varepsilon_+(t) = \varepsilon Z_+([t/\varepsilon]),\ t \in [0,T])$ satisfies SPLD principle in $D([0,T],\mathbb{R}^d)$ with a good rate function

$$I^+_{[0,T]}(\phi) \ = \ \begin{cases} \int_0^T (\log\varphi)^*(\dot\phi(t))\,dt, & \text{if } \phi \text{ is absolutely continuous and} \\ & \ \phi(t) \in \mathbb{R}^{d-1}\times\mathbb{R}_+ \text{ for all } t \in [0,T], \\ +\infty & \text{otherwise} \end{cases}$$

where $(\log\varphi)^*$ denotes the convex conjugate of the function $\log\varphi$:

$$(\log\varphi)^*(v) \doteq \sup_{a\in\mathbb{R}^d}\Big(a\cdot v - \log\varphi(a)\Big).$$

The next proposition provides the SPLD principle for the scaled processes $Z^\varepsilon(t)$.

Proposition 5.1. *Under the hypotheses $(H_0)-(H_4)$, for every $T>0$, the family of scaled processes $(Z^\varepsilon([t/\varepsilon]) = \varepsilon Z([t/\varepsilon]),\, t\in[0,T])$ satisfies SPLD principle in $D([0,T],\mathbb{R}^d)$ with a good rate function*

$$I_{[0,T]}(\phi) \;=\; \begin{cases} \int_0^T L(\phi(t),\dot\phi(t))\,dt, & \text{if } \phi \text{ is absolutely continuous and} \\ & \phi(t)\in\mathbb{R}^{d-1}\times\mathbb{R}_+ \text{ for all } t\in[0,T], \\ +\infty & \text{otherwise.}\end{cases}$$

The local rate function $L(z,v)$ is defined for every $z=(x,y), v\in\mathbb{R}^{d-1}\times\mathbb{R}$ by the equality

$$L(z,v) \;=\; \begin{cases}(\log\varphi)^*(v) & \text{if } y>0\\ (\log\ \max\{\varphi,\varphi_0\})^*(v) & \text{if } y=0\end{cases}$$

where $(\log\max\{\varphi,\varphi_0\})^*$ is the convex conjugate of the function $\log\max\{\varphi,\varphi_0\}$:

$$(\log\max\{\varphi,\varphi_0\})^*(v) \;=\; \sup_{a\in\mathbb{R}^d}\ (a\cdot v - \log\ \max\{\varphi(a),\varphi_0(a)\})\,.$$

This proposition is a consequence of the results obtained in [8, 10, 13, 14]. The results of Dupuis, Ellis and Weiss [8] prove that $I_{[0,T]}$ is a good rate function on $D([0,T],\mathbb{R}^d)$ and provide the SPLD upper bound. Because of the communication condition, SPLD lower bound follows from the local estimates obtained in [13], the general SPLD lower bound of Dupuis and Ellis [10] and the integral representation of the corresponding rate function obtained in [14]. For the related results, see also [2, 9, 16, 22].

5.2. **Explicit form of quasi-potentials.** For a given rate function $J_{[0,T]}$ on the Skorohod space $D([0,T],\mathbb{R}^d)$, the quantity

$$J(q,q') \;=\; \inf_{T>0}\ \inf_{\substack{\phi\in D([0,T],\mathbb{R}^d):\\ \phi(0)=q,\phi(T)=q'}} J_{[0,T]}(\phi)$$

represents the optimal large deviation cost to go from q to q'. Following Freidlin and Wentzel terminology [11], such a function $I:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}_+$ is called *quasi-potential*. Borovkov and Mogulskii [4] called this function *second deviation rate function*.

In this section, we calculate explicitly the quasi-potentials $I(0,q)$ and $I^+(q',q)$ of the rate functions $I_{[0,T]}$ and $I^+_{[0,T]}$ respectively.

Proposition 5.2. *Under the hypotheses (H2) and (H4), for any $q',q\in\mathbb{R}^{d-1}\times\mathbb{R}_+$,*

$$I^+(q',q) \;=\; \sup_{a\in D} a\cdot(q-q'). \tag{5.3}$$

Proof. Indeed, for any $T>0$ and any absolutely continuous function $\phi:[0,T]\to\mathbb{R}^{d-1}\times\mathbb{R}_+$ with $\phi(0)=q'$ and $\phi(T)=q$,

$$I^+_{[0,T]}(\phi) \;=\; \int_0^T (\log\varphi)^*(\dot\phi(t))\,dt \;\geq\; T(\log\varphi)^*\left(\frac{q-q'}{T}\right)$$

because the function $(\log\varphi)^*$ is convex. Since the last relation holds with the equality for the linear function $\phi(t) = t(q-q')/T$, $t \in [0,T]$ we obtain

$$I^+(0,q) \;=\; \inf_{T>0} T(\log\varphi)^* \left(\frac{q-q'}{T}\right). \tag{5.4}$$

Furthermore, under the hypotheses (H2) and (H4), the function $\log\varphi$ is convex and continuous on $\mathbb{R}^d$ and hence, it is a closed convex proper function on $\mathbb{R}^d$. By Theorem 13.5 of Rockafellar [19] from this it follows that the support function of the set $D = \{a \in \mathbb{R}^d : \ \log\varphi(a) \le 0\}$ is equal to the closure of the positively homogeneous convex function k generated by $(\log\varphi)^*$. For any $v \in \mathbb{R}^d$ we have therefore

$$\mathrm{cl}(k)(v) \;=\; \sup_{a\in D} a\cdot v$$

Moreover, under the hypotheses (H2) and (H4), $(\log\varphi)^*$ is also a closed convex proper function on $\mathbb{R}^d$ with

$$0 \;<\; (\log\varphi)^*(0) \doteq -\inf_{a\in\mathbb{R}^d}\varphi(a) \;<\; +\infty.$$

By Theorem 9.7 of Rockafellar [19] from this it follows that the positively homogeneous convex function k generated by $(\log\varphi)^*$ is closed and for any $q'q \in \mathbb{R}^{d-1}\times\{0\}$, the quantity $k(q-q')$ is equal to the right hand side of (5.4). Hence, for any $q', q \in \mathbb{R}^{d-1}\times\{0\}$,

$$I^+(q',q) \;=\; k(q-q') \;=\; \mathrm{cl}(k)(q-q') \;=\; \sup_{a\in D} a\cdot(q-q').$$

Proposition 5.2 is therefore proved. □

The next proposition identifies the quasi-potential of the rate function $I_{[0,T]}$.

Proposition 5.3. *Under the hypotheses (H0)-(H4), for any non-zero vector $q \in \mathbb{R}^{d-1}\times\mathbb{R}_+$,*

$$I(0,q) \;=\; \inf_{\gamma\in\mathbb{R}^{d-1}\times\{0\}} I(0,\gamma) + I^+(\gamma,q) \;=\; \sup_{a\in(\Theta\times\mathbb{R})\cap D} a\cdot q \tag{5.5}$$

Proof. Indeed, the first equality of (5.5) holds because for any absolutely continuous function $\phi : [0,T]\to\mathbb{R}^{d-1}\times\mathbb{R}_+$ with $\tau = \sup\{t>0 : \phi(t)\in\mathbb{R}^{d-1}\times\{0\}\}$, one has

$$\begin{aligned} I_{[0,T]}(\phi) \;&=\; \int_0^T L(\phi(t),\dot\phi(t))\,dt \;=\; \int_0^\tau L(\phi(t),\dot\phi(t))\,dt + \int_\tau^T (\log\varphi)^*(\dot\phi(t))\,dt \\ &=\; I_{[0,\tau]}(\phi) + I^+_{[0,T-\tau]}(\phi_\tau) \end{aligned}$$

where $\phi : [0,\tau]\to\mathbb{R}^{d-1}\times\mathbb{R}_+$ is the restriction of the function ϕ on $[0,\tau]$ and $\phi_\tau : [0,T-\tau]\to\mathbb{R}^{d-1}\times\mathbb{R}_+$ is defined by $\phi_\tau(t) = \phi(\tau+t)$ for all $t\in[0,T-\tau]$. To get the second equality of (5.5) we first notice that for any $T>0$ and any absolutely continuous function $\phi : [0,T]\to\mathbb{R}^{d-1}\times\mathbb{R}_+$ with $\phi(0)=0$ and $\phi(T) = \gamma\in\mathbb{R}^{d-1}\times\{0\}$, the following relations hold

$$\begin{aligned} I_{[0,T]}(\phi) = \int_0^T L(\phi(t),\dot\phi(t))\,dt &\ge \int_0^T (\log\ \max\{\varphi,\varphi_0\})^*(\dot\phi(t))\,dt \\ &\ge\; T(\log\ \max\{\varphi,\varphi_0\})^*\left(\frac{\gamma}{T}\right). \end{aligned}$$

The first inequality holds here because according to the definition of the local rate function,

$$L(x,v) \;\geq\; (\log\max\{\varphi,\varphi_0\})^*(v), \quad \forall v\in\mathbb{R}^d,\; x\in\mathbb{R}^{d-1}\times\mathbb{R}_+.$$

The second inequality is satisfied because the function $(\log\ \max\{\varphi,\varphi_0\})^*$ is convex. Since these relations hold with the equalities for the linear function $\phi(t) = t\gamma/T$, we obtain

$$I(0,\gamma) \;=\; \inf_{T>0} T(\log\ \max\{\varphi,\varphi_0\})^*\left(\frac{\gamma}{T}\right), \quad \forall\gamma\in\mathbb{R}^{d-1}\times\{0\},$$

and using next the same arguments as in the proof of Proposition 5.2 we conclude that

$$I(0,\gamma) \;=\; \sup_{a:\ \max\{\varphi(a),\varphi_0(a)\}\leq 1} a\cdot\gamma \;=\; \sup_{a\in D\cap D_0} a\cdot\gamma, \qquad \forall\gamma\in\mathbb{R}^{d-1}\times\{0\}.$$

From the last relation it follows that

$$I(0,\gamma) \;=\; \sup_{a\in\Theta\times\mathbb{R}} a\cdot\gamma, \qquad \forall\gamma\in\mathbb{R}^{d-1}\times\{0\}$$

because the set $\Theta\times\{0\}$ is the orthogonal projection of the set $D\cap D_0$ onto the hyper-plane $\mathbb{R}^{d-1}\times\{0\}$. Using Proposition 5.3 we obtain therefore

$$\inf_{\gamma\in\mathbb{R}^{d-1}\times\{0\}} I(0,\gamma)+I^+(\gamma,q) \;=\; \inf_{\gamma\in\mathbb{R}^{d-1}\times\{0\}}\left(\sup_{a\in\Theta\times\mathbb{R}} a\cdot\gamma + \sup_{a\in D} a\cdot(q-\gamma)\right)$$

Moreover, since for $\gamma\in\mathbb{R}^d$ with a non-zero last coordinate on has

$$\sup_{a\in\Theta\times\mathbb{R}} a\cdot\gamma \;=\; +\infty,$$

the infimum over $\gamma\in\mathbb{R}^{d-1}\times\{0\}$ at the right hand side of the above relation can be replaced by the infimum over $\gamma\in\mathbb{R}^d$. Finally, under the hypotheses (H0)-(H4), the interior of the set $\hat{D} = (\Theta\times\mathbb{R})\cap D$ is non-empty and consequently, by Corollary 16.4.1 of Rockafellar [19],

$$\inf_{\gamma\in\mathbb{R}^d}\left(\sup_{a\in\Theta\times\mathbb{R}} a\cdot\gamma + \sup_{a\in D} a\cdot(q-\gamma)\right) \;=\; \sup_{a\in(\Theta\times\mathbb{R})\cap D} a\cdot q.$$

The second equality of (5.5) is therefore proved. □

Corollary 5.1. *Under the hypotheses (H0) - (H4), the functions $q\to I^+(0,q)$ and $q\to I(0,q)$ are convex and continuous everywhere on $\mathbb{R}^{d-1}\times\mathbb{R}_+$.*

Proof. Indeed, the equalities (5.3) and (5.5) show that each of these functions is a support function of a compact set. From this it follows that they are finite, convex and therefore continuous on $\mathbb{R}^{d-1}\times\mathbb{R}_+$. □

The next proposition investigates the point where the minimum of the function $\gamma\to I(0,\gamma)+I^+(\gamma,q)$ over $\gamma\in\mathbb{R}^{d-1}\times\{0\}$ is attained. Recall that by Corollary 2.5, for every $q\in\mathbb{R}^{d-1}\times\mathbb{R}_+^*$, there exists a unique point $\gamma_q\in\mathbb{R}^{d-1}\times\{0\}$ for which the vectors γ_q, $q-\gamma_q$ are normal to the set $\hat{D} = (\Theta\times\mathbb{R})\cap D$ at the point $\hat{a}(q)$ and the vector $q-\gamma_q$ is normal to the set D at the point $\hat{a}(q)$.

Proposition 5.4. *For $q\in\mathbb{R}^{d-1}\times\mathbb{R}_+^*$, the point γ_q is the only minimum of the function $\gamma\to I(0,\gamma)+I^+(\gamma,q)$ on the hyperplane $\mathbb{R}^{d-1}\times\{0\}$.*

Proof. Indeed, by Corollary 5.1, the functions $\gamma \to I(0,\gamma)$ and $\gamma \to I^+(\gamma, q) = I^+(0, q-\gamma)$ are finite and convex everywhere on $\mathbb{R}^{d-1} \times \{0\}$. The function $\gamma \to I(0,\gamma) + I^+(\gamma, q)$ is therefore also finite and convex everywhere on $\mathbb{R}^{d-1} \times \{0\}$. By Theorem 23.5 of Rockafellar [19] from this it follows that $I(0,\gamma) + I^+(\gamma, q)$ achieves its minimum over $\mathbb{R}^{d-1} \times \{0\}$ at the point $\hat\gamma \in \mathbb{R}^{d-1} \times \{0\}$ if and only if the differential $\partial(I(0,\hat\gamma) + I^+(\hat\gamma, q))$ of this function at the point $\hat\gamma$ contains zero vector. Theorem 23.8 of Rockafellar [19] proves that

$$\partial(I(0,\hat\gamma) + I^+(\hat\gamma, q)) \; = \; \partial I(0,\hat\gamma) + \partial I^+(\hat\gamma, q)$$

where $\partial I(0,\hat\gamma)$ denotes the differential of the function $\gamma \to I(0,\gamma)$ and $\partial I^+(\hat\gamma, q)$ is the differential of the function $\gamma \to I^+(\gamma, q)$ at the point $\gamma = \hat\gamma$. By Corollary 23.5.3 of Rockafellar [19], from (5.5) it follows that $\hat a \in \partial I(0,\gamma)$ if and only if $\hat a \in D$ and

$$I(0,\gamma) \; = \; \sup_{a\in(\Theta\times\mathbb{R})\cap D} a\cdot\gamma \; = \; \hat a\cdot\gamma$$

or equivalently, when the vector γ is normal to the set $(\Theta\times\mathbb{R})\cap D$ at the point $\hat a \in (\Theta\times\mathbb{R})\cap D$. Similarly, from (5.3) it follows that $a' = -\hat a \in \partial I^+(\gamma, q)$ if and only if $\hat a \in D$ and

$$I^+(\gamma, q) \; = \; \sup_{a\in(\Theta\times\mathbb{R})\cap D} a\cdot(q-\gamma) \; = \; \hat a\cdot(q-\gamma)$$

or equivalently, when the vector $(q-\gamma)$ is normal to the set D at the point $\hat a \in D$. According to the definition of γ_q, this proves that the function $\gamma \to I(0,\gamma) + I^+(\gamma, q)$ achieves its minimum over the set $\mathbb{R}^{d-1} \times \{0\}$ at the point $\gamma_q \in \mathbb{R}^{d-1} \times \{0\}$.

Conversely, if the function $\gamma \to I(0,\gamma) + I^+(\gamma, q)$ achieves its minimum over the set $\mathbb{R}^{d-1} \times \{0\}$ at some point $\hat\gamma \in \mathbb{R}^{d-1} \times \{0\}$ then there is a point $\hat a \in (\Theta\times\mathbb{R})\cap D$ for which the following conditions are satisfied :

- the vector $\hat\gamma$ is normal to the set $(\Theta\times\mathbb{R})\cap D$ at the point $\hat a$,
- the vector $(q-\hat\gamma)$ is normal to the set D at the point $\hat a$,
- and $I(0,\hat\gamma) + I^+(\hat\gamma, q) \; = \; \hat a\cdot\hat\gamma + \hat a(q-\gamma) \; = \; \hat a\cdot q$.

Moreover, from (5.5) it follows that

$$I(0,\hat\gamma) + I^+(\hat\gamma, q) \; = \; \sup_{a\in(\Theta\times\mathbb{R})\cap D} a\cdot q$$

and consequently,

$$\sup_{a\in(\Theta\times\mathbb{R})\cap D} a\cdot q \; = \; \hat a\cdot q$$

The last relation shows that the vector q is normal to the set $(\Theta\times\mathbb{R})\cap D$ at the point $\hat a \in (\Theta\times\mathbb{R})\cap D$. By Lemma 2.3 from this it follows that $\hat a = \hat a(q)$. The vectors γ_q, $q-\gamma_q$ are therefore normal to the set $\hat D = (\Theta\times\mathbb{R})\cap D$ at the point $\hat a(q)$ and the vector $q-\gamma_q$ is normal to the set D at the point $\hat a(q)$. By Corollary 2.5 this proves that $\hat\gamma = \gamma_q$. □

5.3. **Logarithmic asymptotics of Green's function.** Now, we obtain logarithmic asymptotics of Green's functions $G(z,z')$ and $G_+(z,z')$ for the Markov processes $(Z(t))$ and $(Z_+(t))$.

Proposition 5.5. *Under the hypotheses (H2)-(H4), for any $q \in \mathbb{R}^{d-1}\times\mathbb{R}_+$ and any sequences $\varepsilon_n > 0$ and $z_n \in \mathbb{Z}^{d-1}\times\mathbb{N}^*$ with $\lim_n \varepsilon_n = 0$ and $\lim_n \varepsilon_n z_n = q$ the following relations hold*

$$\lim_{n\to\infty} \varepsilon_n \log G_+(z, z_n) \; = \; -\sup_{a\in D} a\cdot q.$$

Proof. Indeed, let the sequences $\varepsilon_n > 0$ and $z_n \in \mathbb{Z}^{d-1}\times\mathbb{N}^*$ be such that $\lim_n \varepsilon_n = 0$ and $\lim_n \varepsilon_n z_n = q$. Then by Lemma 2.1, for any $a \in D$,

$$G_+(z,z_n) \;\leq\; \exp(a\cdot(z-z_n))G_S(0,0), \qquad \forall n\in\mathbb{N}, z\in\mathbb{Z}^{d-1}\times\mathbb{N}^*$$

and consequently,

$$\lim_{n\to\infty} \varepsilon_n \log G_+(z,z_n) \;\leq\; -a\cdot q \qquad \forall a\in D$$

from which it follows that

$$\lim_{n\to\infty} \varepsilon_n \log G_+(z,z_n) \;\leq\; -\sup_{a\in D} a\cdot q.$$

The inequality

$$\lim_{n\to\infty} \varepsilon_n \log G_+(z,z_n) \;\geq\; -\sup_{a\in D} a\cdot q$$

was proved in Proposition 4.2 of Ignatiouk [15] by using lower large deviation bound for the scaled processes $Z_+^\varepsilon(t) = \varepsilon Z_+(t/\varepsilon)$ and communication condition. This proof is quite similar to the proof of the lower bound (5.6) below. □

Proposition 5.6. *Under the hypotheses (H0)-(H4), for any $q\in\mathbb{R}^{d-1}\times\mathbb{R}_+$, and any sequences $\varepsilon_n > 0$ and $z_n\in\mathbb{Z}^{d-1}\times\mathbb{N}$ with $\lim_n \varepsilon_n = 0$, and $\lim_{n\to\infty}\varepsilon_n z_n = q$ the following relation holds :*

$$\lim_{n\to\infty} \varepsilon_n \log G(z,z_n) \;=\; -\sup_{a\in(\Theta\times\mathbb{R})\cap D} a\cdot q, \qquad \forall z\in\mathbb{Z}^{d-1}\times\mathbb{N}.$$

Proof. Let two sequences $\varepsilon_n > 0$ and $z_n\in\mathbb{Z}^{d-1}\times\mathbb{N}$ be such that $\lim_n \varepsilon_n = 0$ and $\lim_n \varepsilon_n z_n = q$. We begin our analysis with the proof of the lower bound

$$\lim_{n\to\infty} \varepsilon_n \log G(z,z_n) \;\geq\; -\sup_{a\in(\Theta\times\mathbb{R})\cap D} a\cdot q. \tag{5.6}$$

For this we use the lower large deviation bound and communication condition. Denote for $B\in\mathbb{R}^d$

$$G(z,B) \;=\; \sum_{z'\in B\cap\mathbb{Z}^{d-1}\times\mathbb{N}} G(z,z').$$

The large deviation lower bound implies that for any $\delta > 0$ and $T > 0$,

$$\begin{aligned}
\liminf_{n\to\infty}\; \varepsilon\log G(z,\varepsilon^{-1}B(q,\delta)) \;&\geq\; \liminf_{n\to\infty}\; \varepsilon\log \mathbb{P}_z\left(Z_\varepsilon(T)\in B(q,\delta)\right)\\
&\geq\; -\inf_{\phi\in D([0,T],\mathbb{R}^{d-1}\times\mathbb{R}_+):\ \phi(0)=0,\ \phi(T)\in B(q,\delta)} I_{[0,T]}(\phi)\\
&\geq\; -\inf_{\phi\in D([0,T],\mathbb{R}^{d-1}\times\mathbb{R}_+):\ \phi(0)=0,\ \phi(T)=q} I_{[0,T]}(\phi)
\end{aligned}$$

from which it follows that

$$\lim_{\delta\to 0}\;\liminf_{n\to\infty}\; \varepsilon\log G(z,\varepsilon^{-1}B(q,\delta)) \;\geq\; -I(0,q) = -\sup_{a\in(\Theta\times\mathbb{R})\cap D} a\cdot q \tag{5.7}$$

where the last relation is proved by Proposition 5.3. Moreover, by Proposition 4.1, the Markov process $(Z(t))$ satisfies communication condition and hence, there are $\theta > 0$ and $C > 0$ such that for any $z', z''\in\mathbb{Z}^{d-1}\times\mathbb{N}$ such that $z'\neq z''$, the probability that the Markov process $(Z(t))$ starting at z' hits z'' before the first return to z' is greater than $\theta^{C|z''-z'|}$. This proves that for any $z,z'\in\mathbb{Z}^{d-1}\times\mathbb{N}$ and $n\in\mathbb{N}$

$$G(z,z_n) \;\geq\; G(z,z')\theta^{C|z_n-z'|}$$

and consequently, for all those $n \in \mathbb{N}$ for which $|q - \varepsilon_n z_n| < \delta$, we obtain

$$\begin{aligned} G(z, \varepsilon_n^{-1} B(q,\delta))\ \theta^{2C\delta/\varepsilon_n} &\le \sum_{z' \in \mathbb{Z}^{d-1}\times\mathbb{N}:\ z' \in \varepsilon_n^{-1}B(q,\delta)} G(z,z')\theta^{C|z_n - z'|} \\ &\le \mathrm{Card}\{z \in \mathbb{Z}^d : z \in \varepsilon_n^{-1}B(q,\delta)\}\ G(z,z_n) \\ &\le (2\delta\varepsilon_n^{-1} + 1)^d\ G(z,z_n) \end{aligned}$$

The last inequality shows that

$$\lim_{n\to\infty} \varepsilon_n \log G(z,z_n) \ \ge\ 2C\delta\log\theta + \liminf_{\varepsilon\to 0}\ \varepsilon\ \log G\big(z, \varepsilon^{-1}B(q,\delta)\big)$$

and hence, letting $\delta \to 0$ and using (5.7), we get (5.6)

To prove the inequality

$$\lim_{n\to\infty} \varepsilon_n \log G(z,z_n) \ \le\ -\sup_{a\in(\Theta\times\mathbb{R})\cap D} a\cdot q \tag{5.8}$$

we use Lemma 2.1 and Corollary 2.2. For $a \in D$, $z \in \mathbb{Z}^{d-1}\times\mathbb{N}$ and $z' \in \mathbb{Z}^{d-1}\times\{0\}$, by Lemma 2.1,

$$G(z,z') \ \le\ G(z',z')\exp(\overline{a}\cdot z - \overline{a}\cdot z') \ =\ G(0,0)\exp(\overline{a}\cdot z - a\cdot z').$$

Moreover, if $\varphi_0(\overline{a}) < 1$ then for $z \in \mathbb{Z}^{d-1}\times\mathbb{N}$ and $z' \in \mathbb{Z}^{d-1}\times\mathbb{N}^*$, by Corollary 2.2,

$$\frac{G(z,z')}{G_S(0,0)} \le \exp(a\cdot(z-z')) + \varphi_0(a)(1-\varphi_0(\overline{a}))^{-1}\exp(\overline{a}\cdot z - a\cdot z').$$

These inequalities show that for any $a \in D$ for which $\varphi_0(\overline{a}) < 1$, one has

$$\lim_{n\to\infty} \varepsilon_n \log G(z,z_n) \ \le\ -a\cdot q.$$

The last relation proves (5.8) because $(\Theta\times\mathbb{R})\cap D = \{a \in D : \varphi_0(\overline{a}) \le 1\}$. $\square$

6. Principal part of the renewal equation

Recall that the transition probabilities $p(z,z')$ of the Markov process $(Z(t))$ are the same as transition probabilities $p(z,z') = \mu(z'-z)$ of the homogeneous random walk $S(t)$ on $\mathbb{Z}^d$ for $z \in \mathbb{Z}^{d-1}\times\mathbb{N}^*_+$ and that $p(z,z') = \mu_0(z'-z)$ for $z \in \mathbb{Z}^{d-1}\times\{0\}$. From this it follows that the Green's function $G(z,z')$ satisfies the following renewal equation

$$G(z,z_n) \ =\ G_+(z,z_n) \ +\ \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\} \\ w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*}} G(z,w)\mu_0(w'-w)G_+(w',z_n). \tag{6.1}$$

$G_+(z,z')$ denotes here Green's function of the homogeneous random walk $(Z_+(t))$ killed upon hitting the half-space $\mathbb{Z}^{d-1}\times(-\mathbb{N})$:

$$G_+(z,z') \doteq \sum_{t=0}^{\infty} \mathbb{P}_z(Z_+(t) = z') \doteq \sum_{t=0}^{\infty} \mathbb{P}_z(S(t) = z';\ \tau > t)$$

where $\tau = \inf\{t \ge 1 : S(t) \in \mathbb{Z}^{d-1}\times\{0\}\}$.

In this section we show that for a sequence $z_n \in \mathbb{Z}^{d-1}\times\mathbb{N}$ with $\lim_n |z_n| = \infty$ and $\lim_n z_n/|z_n| = q \in \mathbb{R}^{d-1}\times]0,+\infty[$ the right hand side of the renewal equation (6.1) can be decomposed into a main part

$$\Xi^q_\delta(z,z_n) \ =\ G_+(z,z_n)\ 1_{\{\gamma_q = 0\}} + \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\}, w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*: \\ |w-\gamma_q|z_n||\le\delta|z_n|}} G(z,w)\mu_0(w'-w)G_+(w',z_n)$$

and the corresponding negligible part $G(z,z_n) - \Xi^q_\delta(z,z_n)$. Recall that γ_q is a only vector on the boundary hyperplane $\mathbb{R}^{d-1}\times\{0\}$ for which the vectors γ_q and $q-\gamma_q$ belong to the normal cone $V(\hat a(q))$ to the set $\hat D \doteq \{a\in D : \varphi'\overline{a}) \le 1\} = (\Theta\times\mathbb{R})\cap D$ at the point $\hat a(q)$ and the vector $q-\gamma_q$ is normal to the set D at the point $\hat a(q)$ (see Corollary 2.4). By Propositions 5.3 and 5.4, this is also the only minimum of the function $\gamma\to I(0,\gamma)+I^+(\gamma,q)$ on the boundary hyperplane $\mathbb{R}^{d-1}\times\{0\}$ where

$$I(0,q) \;=\; I(0,\gamma_q) + I^+(\gamma_q,q).$$

We begin our analysis we the following lemma.

Lemma 6.1. *Under the hypotheses (H0)-(H4),*

$$I_{min} \doteq \inf_{\gamma\in\mathbb{R}^{d-1}\times\{0\}:\ |\gamma|=1} I(0,\gamma)+I^+(\gamma,0) \;>\; 0$$

Proof. Indeed, by Corollary 5.1, the function

$$\gamma\to I(0,\gamma)+I^+(\gamma,0) = I(0,\gamma)+I^+(0,-\gamma)$$

is continuous. To prove our lemma it is therefore sufficient to show that

$$I(0,\gamma)+I^+(0,-\gamma) > 0, \qquad \forall\,\gamma\neq 0. \tag{6.2}$$

To prove this inequality let us notice that

$$\begin{aligned} I(0,\gamma)+I^+(0,-\gamma) &= \sup_{a\in\hat D} a\cdot\gamma + \sup_{a\in D} a\cdot(-\gamma) \\ &\ge \sup_{a\in\hat D} a\cdot\gamma + \sup_{a\in\hat D} a\cdot(-\gamma) \;=\; \sup_{a\in\hat D} a\cdot\gamma \;-\; \inf_{a\in\hat D} a\cdot\gamma \;\ge\; 0 \end{aligned}$$

where the last relation holds with equality if and only if $a\cdot\gamma = 0$ for all $a\in\hat D$. Under the hypotheses of our lemma, for any non-zero vector $\gamma\in\mathbb{R}^d$ there is $a\in\hat D$ for which $a\cdot\gamma\neq 0$ because the set $\hat D$ has a non-empty interior (see the proof of Lemma 2.5). The inequality (6.2) is therefore proved. $\square$

Proposition 6.1. *Under the hypotheses (H0)-(H4), for any $q\in\mathcal{S}^d_+\cap\mathbb{R}^{d-1}\times]0,+\infty[$ $\delta>0$ and any sequence $z_n\in\mathbb{Z}^{d-1}\times\mathbb{N}$ with $\lim_n|z_n|=\infty$ and $\lim_n z_n/|z_n| = q$,*

$$\lim_{n\to\infty} \Xi^q_\delta(z,z_n)/G(z,z_n) \;=\; 1, \qquad \forall z\in\mathbb{Z}^{d-1}\times\mathbb{N}. \tag{6.3}$$

Proof. Let $q\in\mathcal{S}^d_+\cap\mathbb{R}^{d-1}\times]0,+\infty[$ and let a sequence $z_n\in\mathbb{Z}^{d-1}\times\mathbb{N}$ be such that $|z_n|\to\infty$ and $z_n/|z_n|\to q$ as $n\to\infty$. Then by Proposition 5.4,

$$\lim_{n\to\infty}\frac{1}{|z_n|}\log G(z,z_n) \;=\; -\,I(0,q),$$

and hence, to get (6.3) it is sufficient to show that

$$\limsup_{n\to\infty}\frac{1}{|z_n|}\log(G(z,z_n)-\Xi^q_\delta(z,z_n)) \;<\; -\,I(0,q).$$

By Lemma 1.2.15 of Dembo and Zeitouni [5], for this it is sufficient to prove the following three inequalities :

$$\limsup_{n\to\infty}\frac{1}{|z_n|}\log G_+(z,z_n) < -I(0,q) \quad\text{ when }\quad \gamma_q\neq 0, \tag{6.4}$$

$$\limsup_{n\to\infty}\frac{1}{|z_n|}\;\log\sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\},w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*:\\ |w'-w|>\delta'|z_n|}} G(z,w)\mu_0(w'-w)G_+(w',z_n) \;<\; -I(0,q) \tag{6.5}$$

and
(6.6)

$$\limsup_{n\to\infty} \frac{1}{|z_n|} \log \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\}, w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*: \\ |w-\gamma_q|z_n||>\delta|z_n|,\ |w'-w|\le\delta'|z_n|}} G(z,w)\mu_0(w'-w)G_+(w',z_n) \ < \ -I(0,q)$$

for some $\delta' > 0$ small enough.

Proof of (6.4)*:* This relation is a consequence of Propositions 5.3, 5.4 and 5.5. Namely, Proposition 5.5 proves that

$$\lim_{n\to\infty} \frac{1}{|z_n|} \log G_+(z,z_n) \ = \ - \ I^+(0,q)$$

and by Propositions 5.3 and 5.4,

$$I^+(0,q) \ = \ I(0,0) + I^+(0,q) \ > \ I(0,\gamma_q) + I^+(\gamma_q,q) \ = \ I(0,q) \quad \text{when} \quad \gamma_q \neq 0.$$

Proof of (6.5)*:* For any $a \in D$ for which $\varphi_0(a) < 1$, with the same arguments as in the proof of Corollary 2.2 one gets

$$\begin{aligned}
&\sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\}, w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*: \\ |w'-w|>\delta'|z_n|}} G(z,w)\mu_0(w'-w)G_+(w',z_n)\exp(a\cdot z_n) \\
&\qquad\qquad \le \ G_S(0,0) \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\}, w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*: \\ |w'-w|>\delta'|z_n|}} G(z,w)\mu_0(w'-w)\exp(a\cdot w') \\
&\qquad\qquad \le \ \frac{G_S(0,0)\exp(\overline{a}\cdot z)}{1-\varphi_0(\overline{a})} \sum_{u\in\mathbb{Z}^{d-1}\times\mathbb{N}^*:\ |u|>\delta'|z_n|} \mu_0(u)\exp(a\cdot u).
\end{aligned}$$

Hence, the right hand side of (6.5) does not exceed

$$-\lim_{n\to\infty} a\cdot z_n/|z_n| + \delta' \limsup_{R\to\infty} \frac{1}{R} \log \sum_{u\in\mathbb{Z}^{d-1}\times\mathbb{N}:\ |u|>R} \mu_0(u)\exp(a\cdot u)$$

where

$$\limsup_{R\to\infty} \frac{1}{R} \log \sum_{z\in\mathbb{Z}^{d-1}\times\mathbb{N}:\ |z|>R} \mu_0(z)\exp(a\cdot z) \ = \ -\infty$$

because under the hypotheses (H4), the function $a' \to \varphi_0(a+a')$ is finite everywhere in $\mathbb{R}^d$. Relation (6.5) is therefore proved.

Proof of (6.6)*:* Lemma 2.1 proves that for any $w \in \mathbb{Z}^{d-1}\times\{0\}$, $w' \in \mathbb{Z}^{d-1}\times\mathbb{N}^*$ and $a, a' \in D$ with $\varphi_0(\overline{a}) \le 1$ and $|w'-w| \le \delta'|z_n|$,

$$G(z,w)G_+(w',z_n) \ \le \ G(w,w)G_S(0,0)\exp(\overline{a}\cdot(z-w) + a'\cdot(w'-z_n))$$

where $G(w,w) = G(0,0)$ and

$$\begin{aligned}
\overline{a}\cdot z + a'\cdot(w'-z_n) &= \overline{a}\cdot z + a'\cdot(w-q|z_n|) + a'(q|z_n|-z_n) + a'(w'-w) \\
&\le c|z| + c\big|q|z_n| - z_n\big| + \delta' c|z_n|
\end{aligned}$$

with $c = \max_{a\in D}|a|$. Moreover, according to the definition of the mapping $a \to \overline{a}$,

$$\overline{a}\cdot w = a\cdot w$$

because $w \in \mathbb{Z}^{d-1} \times \{0\}$ and consequently,

$$G(z,w)G_+(w',z_n) \le \; G(0,0)G_S(0,0)\exp(-a\cdot w - a'\cdot(q|z_n| - w)) \\ \times \exp(c|z_n - q|z_n|| + c|z| + \delta' c|z_n|)$$

Since the last inequality holds for arbitrary $a' \in D$ and $a \in \hat{D} \doteq \{a \in D : \varphi(\overline{a}) \le 1\}$, using Propositions 5.2 and 5.3 we get

$$G(z,w)G_+(w',z_n) \le G(0,0)G_S(0,0)\exp(-I(0,w) - I^+(w, q|z_n|)) \\ \times \exp(c|z_n - q|z_n|| + c|z| + \delta' c|z_n|)$$

from which it follows that

$$\sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\}, w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*: \\ \delta|z_n|<|w-\gamma_q|z_n||\le R|z_n|,\ |w'-w|\le\delta'|z_n|}} G(z,w)\mu_0(w'-w)G_+(w',z_n)$$
$$\le \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\}: \\ \delta|z_n|<|w-\gamma_q|z_n||\le R|z_n|,}} G(0,0)G_S(0,0)\exp(-I(0,w) - I^+(w,q|z_n|))$$
$$\times \exp(c|z_n - q|z_n|| + c|z| + \delta' c|z_n|)$$

and consequently,

$$\limsup_{n\to\infty} \frac{1}{|z_n|}\log \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\}, w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*: \\ \delta|z_n|<|w-\gamma_q|z_n||\le R|z_n|,\ |w'-w|\le\delta|z_n|}} G(z,w)\mu_0(w'-w)G_+(w',z_n)$$
$$\le \limsup_{\varepsilon\to 0} \; \varepsilon\log \sum_{w\in\mathbb{Z}^{d-1}\times\{0\}:\ \delta<|\varepsilon w-\gamma_q|\le R,} \exp(-I(0,w) - I^+(w,q/\varepsilon)) + c\delta'$$
$$\le -\inf_{\gamma\in\mathbb{R}^{d-1}\times\{0\}:\delta<|\gamma_q-\gamma|\le R} (I(0,\gamma) + I^+(\gamma,q)) + c\delta'$$

where the last relation holds because the number of points $w \in \mathbb{Z}^{d-1}\times\{0\}$ satisfying the inequality $\delta < |\varepsilon w - \gamma_q| \le R$ does not exceed $(1+2R/\varepsilon)^d$ and for each of them, $I(0,w) + I^+(w,q/\varepsilon) \;=\; \varepsilon^{-1}(I(0,\gamma) + I^+(\gamma,q))$ with $\gamma = \varepsilon w$. Recall now that the function $\gamma \to I(0,\gamma) + I^+(\gamma,q)$ is convex and continuous on $\mathbb{R}^{d-1}\times\{0\}$, the point γ_q is the only minimum of this function at $\mathbb{R}^{d-1}\times\{0\}$ and $I(0,\gamma_q) + I^+(\gamma_q) = I(0,q)$ (see Corollary 5.1 and Propositions 5.3 and 5.4). This proves that

$$\inf_{\substack{\gamma\in\mathbb{R}^{d-1}\times\{0\}: \\ \delta<|\gamma_q-\gamma|\le R}} (I(0,\gamma) + I^+(\gamma,q)) \;\ge\; \inf_{\gamma\in\mathbb{R}^{d-1}\times\{0\}:\ \delta<|\gamma_q-\gamma|} (I(0,\gamma) + I^+(\gamma,q))$$
$$> \; I(0,\gamma_q) + I^+(\gamma_q,q) \;=\; I(0,q).$$

and consequently, for any $R > \delta > 0$ and $\delta' > 0$ satisfying the inequality

$$0 < \delta' c < \inf_{\gamma\in\mathbb{R}^{d-1}\times\{0\}:\ \delta<|\gamma_q-\gamma|} (I(0,\gamma) + I^+(\gamma,q)) - I(0,q)$$

we get

$$\limsup_{n\to\infty} \frac{1}{|z_n|}\log \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\}, w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*: \\ \delta|z_n|<|w-\gamma_q|z_n||\le R|z_n|,\ |w'-w|\le\delta'|z_n|}} G(z,w)\mu_0(w'-w)G_+(w',z_n) \;<\; -I(0,q).$$

Now, to complete the proof of (6.6) it is sufficient to show that there is $R > 0$ such that

(6.7)
$$\limsup_{n\to\infty} \frac{1}{|z_n|} \log \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\},w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*:\\ |w-\gamma_q|z_n||>R|z_n|,\ |w'-w|\le\delta'|z_n|}} G(z,w)\mu_0(w'-w)G_+(w',z_n) \ < \ -I(0,q).$$

To get this inequality we use again Lemma 2.1 combined with Propositions 5.2 and 5.3 : for any $a, a' \in D$, $w \in \mathbb{Z}^{d-1}\times\{0\}$ and $w' \in \mathbb{Z}^{d-1}\times\mathbb{N}^*$ with $\varphi(\overline{a}) \le 1$, and $|w - w'| \le \delta'|z_n|$, from Lemma 2.1 it follows that

$$\begin{aligned} G(z,w)G_+(w',z_n) &\le G(w,w)G_S(0,0)\exp(\overline{a}\cdot(z-w) + a'\cdot(w'-z_n)) \\ &= G(w,w)G_S(0,0)\exp(-a\cdot w + a'\cdot w + \overline{a}\cdot z + a'\cdot(w'-w-z_n)) \\ &\le G(w,w)G_S(0,0)\exp(-a\cdot w + a'\cdot w + c|z| + (1+\delta')c|z_n|) \end{aligned}$$

with $c = \max_{a\in D}|a|$ and $G(w,w) = G(0,0)$. Using therefore Propositions 5.2 and 5.3 we obtain

$$\begin{aligned} G(z,w)G_+(w',z_n) \ \le \ & G(0,0)G_S(0,0)\exp(-I(0,w) - I^+(0,-w)) \\ & \times \exp(c|z| + c(1+\delta')|z_n|) \end{aligned}$$

from which it follows that

$$\begin{aligned} &\sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\},w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*:\\ |w-\gamma_q|z_n||>R|z_n|,\ |w'-w|\le\delta'|z_n|}} G(z,w)\mu_0(w'-w)G_+(w',z_n) \\ &\le \sum_{w\in\mathbb{Z}^{d-1}\times\{0\}:\ |w-\gamma_q|z_n||>R|z_n|,} G(0,0)G_S(0,0)\exp(-I(0,w) - I^+(0,-w)) \\ &\qquad\qquad \times \exp(c|z| + c(1+\delta')|z_n|) \end{aligned}$$

and consequently,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{|z_n|} \log \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\},w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*:\\ |w-\gamma_q|z_n||>R|z_n|,\ |w'-w|\le\delta'|z_n|}} G(z,w)\mu_0(w'-w)G_+(w',z_n) \\ &\le \ \limsup_{\varepsilon\to 0} \ \varepsilon\log \sum_{\gamma\in\varepsilon\mathbb{Z}^{d-1}\times\{0\}:\ |\gamma-\gamma_q|>R} \exp\left(-I(0,\gamma/\varepsilon) - I^+(0,-\gamma/\varepsilon)\right) \ + \ (1+\delta')c. \end{aligned}$$

Remark finally that

$$I(0,\gamma/\varepsilon) + I^+(0,-\gamma/\varepsilon) \ = \ \Big(I(0,\gamma/|\gamma|) + I^+(0,-\gamma/|\gamma|)\Big)|\gamma|/\varepsilon \ \ge \ I_{min}|\gamma|/\varepsilon$$

where by Lemma 6.1,

$$I_{min} \doteq \inf_{\gamma\in\mathbb{R}^{d-1}\times\{0\}:\ |\gamma|=1} I(0,\gamma) + I^+(0,-\gamma) \ > \ 0.$$

This proves that the right hand side of (6.7) does not exceed

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \ \varepsilon\log \sum_{n\ge R-|\gamma_q|} \mathrm{Card}\{\gamma\in\varepsilon\mathbb{Z}^d:\ n\le|\gamma|\le n+1\}\exp(-I_{min}n/\varepsilon) + (1+\delta')c \\ &\le \limsup_{\varepsilon\to 0} \ \varepsilon\log \sum_{n\ge R-|\gamma_q|} (1+2(n+1)/\varepsilon)^{d-1}\exp(-I_{min}n/\varepsilon) \ + \ (1+\delta')c \\ &\le -I_{min}(R-|\gamma_q|) + (1+\delta')c. \end{aligned}$$

The inequality (6.7) holds therefore for $R > |\gamma_q| + ((1+\delta')c + I(0,q))/I_{min}$. □

7. Ratio limit theorem for Markov-additive processes

In this section we recall the ratio limit theorem for Markov-additive processes.

A Markov chain $\mathcal{Z}(t) = (A(t), M(t))$ on $\mathbb{Z}^{d-1}\times\mathbb{N}$ with transition probabilities $p\big((x,y),(x',y')\big)$ is called *Markov-additive* if

$$p\big((x,y),(x',y')\big) \; = \; p\big((0,y),(x'-x,y')\big)$$

for all $x,x'\in\mathbb{Z}^{d-1}$, $y,y'\in\mathbb{N}$. $A(t)$ is an *additive* part of the process $\mathcal{Z}(t)$, and $M(t)$ is its *Markovian part.* The Markovian part $M(t)$ is a Markov chain on $\mathbb{N}$ with transition probabilities

$$p_M(y,y') = \sum_{x\in\mathbb{Z}^{d-1}} p\big((0,y),(x,y')\big).$$

The assumption we need on the Markov-additive process $\mathcal{Z}(t) = (A(t), M(t))$ are the following.

(A1) There exist $\theta > 0$ and $C > 0$ such that for any $z, z'\in\mathbb{Z}^{d-1}\times\mathbb{N}$ there is a sequence of points $z_0, z_1, \dots, z_n \in\mathbb{Z}^{d-1}\times\mathbb{N}$ with $z_0 = z$, $z_n = z'$ and $n\le C|z'-z|$ such that

$$|z_i - z_{i-1}|\le C \quad \text{and} \quad \mathbb{P}_{z_{i-1}}(\mathcal{Z}(1) = z_i)\ge\theta, \qquad \forall\, i = 1,\dots,n.$$

(A2) *The function*

$$\hat\varphi(a) \; = \; \sup_{z\in\mathbb{Z}^{d-1}\times\mathbb{N}} \mathbb{E}_z\big(\exp(a\cdot(\mathcal{Z}(1)-z))\big)$$

is finite everywhere on $\mathbb{R}^d$.

(A3) *Up to multiplication by constants, there is a unique positive harmonic function* h *of the Markov process* $\mathcal{Z}(t) = (A(t), M(t))$ *such that*

$$\sup_{x\in\mathbb{Z}^{d-1}} h(x,y) < \infty. \tag{7.1}$$

Remark that the Markov-additive process $\mathcal{Z}(t) = (A(t), M(t))$ is not necessarily stochastic : in some points $z = (x,y)\in\mathbb{Z}^{d-1}\times\mathbb{N}$, the transition matrix can be strictly sub-stochastic. When the Markov-additive process $\mathcal{Z}(t) = (A(t), M(t))$ is stochastic, the last assumption means that the only positive harmonic functions $h:\mathbb{Z}^{d-1}\times\mathbb{N}\to\mathbb{R}_+$ satisfying (7.1) are constant.

If the assumption (A1) is satisfied then there is a bounded function $n_0:\mathbb{N}\to\mathbb{N}^*$ such that for any $z = (x,y)\in\mathbb{Z}^{d-1}\times\mathbb{N}$,

$$p^{(n_0(y))}\big((x,y),(x,y)\big)\ge\theta^{n_0(y)} > 0$$

and hence, there is $k\in\mathbb{N}^*$ (for instance, $k = n!$ with $n = \max_y n_0(y)$) such that

$$p^{(k)}\big(z,z\big)\ge\theta^k, \qquad \forall z\in\mathbb{Z}^{d-1}\times\mathbb{N}.$$

Let $\hat k$ be the greatest common divisor of the set of all integers $k > 0$ for which

$$\inf_{z\in Z^{d-1}\times\mathbb{N}} p^{(k)}(z,z) \; > \; 0$$

then from (A3) it follows that

(A3') *Up to multiplication by constants, there is a unique positive harmonic function* h *of the Markov process* $\mathcal{Z}(t) = (A(t), M(t))$ *satisfying the equality* $h(z+\hat k w) = h(z)$ *for all* $z\in\mathbb{Z}^{d-1}\times\mathbb{N}$ *and* $w\in\mathbb{Z}^{d-1}\times\{0\}$.

We will use the following property of Markov-additive processes. $G(z,z')$ denotes here Green's function of the Markov process $\mathcal{Z}(t) = (A(t), M(t))$.

Proposition 7.1. *Let a Markov-additive process $\mathcal{Z}(t) = (A(t), M(t))$ be transient and satisfy the hypotheses (A1), (A2), (A3). Suppose moreover that a sequence of points $z_n \in \mathbb{Z}^{d-1}\times\mathbb{N}$ is such that $|z_n| \to \infty$ and*

$$\liminf_{n\to\infty} \; \frac{1}{|z_n|} \log G\big(z_0, z_n\big) \;\geq\; 0.$$

Then

$$\lim_{n\to\infty} G(z,z_n)/G(z',z_n) \;=\; h(z)/h(z')$$

for all $z, z' \in \mathbb{Z}^{d-1}\times E$.

For a Markov-additive processes $\mathcal{Z}(t) = (A(t), M(t))$ with a one-dimensional additive part and for $z_n = (n,y)$ with a given $y\in\mathbb{N}$, this property was obtained by Foley and McDonald [6]. In the present setting, under the hypotheses (A1), (A2) and (A3'), the proof of this proposition is given in [15].

8. Proof of Theorem 1

Under the hypotheses (H1)-(H4), the interior of the set $\hat{D} \doteq \{a\in D: \; \varphi_0(a) \leq 1\}$ is non-empty because $\varphi(0) = \varphi_0(0) = 1$, $\nabla\varphi(0) \;=\; m \;\neq\; 0$ and

$$\frac{\nabla\varphi(0)}{|\nabla\varphi(0)|} + \frac{\nabla\varphi_0(0)}{|\nabla\varphi_0(0)|} \;=\; \frac{m}{|m|} + \frac{m_0}{|m_0|} \;\neq\; 0.$$

From this it follows that $Card\,((\Theta\times\mathbb{R})\cap\partial_+ D) \;>\; 1$ because the orthogonal projection of the set $\{a\in D: \;\varphi_0(a)\leq 1\}$ on the hyper-plane $\mathbb{R}^{d-1}\times\{0\}$ is homeomorphic to the set $(\Theta\times\mathbb{R})\cap\partial_+ D$ where

$$\Theta \doteq \{\alpha\in\mathbb{R}^{d-1}: \; \inf_{\beta\in\mathbb{R}} \max\{\varphi(\alpha,\beta), \varphi_0(\alpha,\beta)\} \leq 1\}.$$

By Proposition 3.1, this proves that there are non-constant non-negative harmonic functions and consequently, by Theorem 6.2 of [21], the Markov process $Z(t)$ is transient. The first assertion of Theorem 1 is therefore proved.

To prove the second assertion we have to show that

$$\lim_{n\to\infty} G(z,z_n)/G(z_0,z_n) = h_{\hat{a}(q)}(z)/h_{\hat{a}(q)}(z_0), \qquad \forall\, z\in\mathbb{Z}^{d-1}\times\mathbb{N}. \tag{8.1}$$

for any non-zero vector $q\in\mathcal{S}^d_+$, and any sequence of points $z_n\in\mathbb{Z}^{d-1}\times\mathbb{N}$ with $\lim_{n\to\infty}|z_n| = +\infty$ and $\lim_{n\to\infty} z_n/|z_n| = q$. The proof of (8.1) is different in each of the following cases :

– Case 1 : $q\in\mathbb{R}^{d-1}\times\{0\}$,
– Case 2 : $q\in\mathbb{R}^{d-1}\times\mathbb{R}^*_+$, and $\varphi_0(\overline{\hat{a}(q)}) < 1$,
– Case 3 : $q\in\mathbb{R}^{d-1}\times\mathbb{R}^*_+$, $\;\varphi_0(\overline{\hat{a}(q)}) = 1$ and $\gamma_q \neq 0$,
– Case 4 : $q\in\mathbb{R}^{d-1}\times\mathbb{R}^*_+$, $\;\varphi_0(\overline{\hat{a}(q)}) = 1$ and $\gamma_q = 0$,

Recall that $a = \hat{a}(q)$ is the only point of the set $(\Theta\times\mathbb{R})\cap\partial_+ D$ for which $q\in V(a)$ (see Lemma 2.5). We denote by $V(a)$ the normal cone to the set $(\Theta\times\mathbb{R})\cap D$ at the point a. By Lemma 2.3, for every $a\in(\Theta\times\mathbb{R})\cap\partial_+ D$,

$$V(a) \;=\; V_D(a) + \big(V_D(\overline{a}) + V_{D_0}(\overline{a})\big)\cap\big(R^{d-1}\times\{0\}\big)$$

where $\overline{a}$ is the only point in the boundary $\partial_- D = \{a \in \partial D : \nabla\varphi(a) \in \mathbb{R}^{d-1} \times \mathbb{R}_-\}$ which has the same orthogonal projection to the hyper-plane as the point a,

$$V_D(a) = \{c\nabla\varphi(a) \mid c \geq 0\}$$

is the normal cone to the set D at the point a and

$$V_{D\cap D_0}(\overline{a}) = V_D(\overline{a}) + V_{D_0}(\overline{a}) = \{c_1\nabla\varphi(\overline{a}) + c_2\nabla\varphi_0(\overline{a}) \mid c_1, c_2 \geq 0\}$$

is the normal cone to the set $D \cap D_0$ at the point $\overline{a}$. For $q \in \mathbb{R}^{d-1} \times \mathbb{R}^*_+$, according to Corollary 2.4,

$$\gamma_q \in \Big(V_D(\overline{\hat{a}(q)}) + V_{D_0}(\overline{\hat{a}(q)})\Big) \cap \left(R^{d-1} \times \{0\}\right)$$

is the only vector at the hyper-plane $\mathbb{R}^{d-1} \times \{0\}$ for which $q - \gamma_q, \gamma_q \in V(\hat{a}(q))$ and $q - \gamma_q \in V_D(\hat{a}(q))$. By Lemma 2.5, for $\gamma_q \neq 0$ we have therefore

$$\hat{a}(q) = \hat{a}(\gamma_q) = \hat{a}(q - \gamma_q). \tag{8.2}$$

Recall finally that for every $a \in (\Theta \times \mathbb{R}) \cap \partial_+ D$,

$$\varphi(\overline{a}) = 1 \quad \text{and} \quad \varphi_0(\overline{a}) \leq 1 \tag{8.3}$$

because $\overline{a} \in \partial_- D \subset \partial D$ according to the definition of the mapping $a \to \overline{a}$, and $\overline{a} \in D_0$ according to the definition of the set Θ.

Case 1 : To get (8.1) in this case we combine the ratio limit theorem and the method of the exponential change of measure : Proposition 7.1 is applied for a twisted Markov process $\tilde{Z}(t)$ on $\mathbb{Z}^{d-1} \times \mathbb{N}$ having transition probabilities

$$\begin{aligned} \tilde{p}(z, z') \;&=\; \exp\left(a \cdot (z' - z)\right) p(z, z') \\ &=\; \begin{cases} \exp\left(a \cdot (z' - z)\right) \mu_0(z' - z) & \text{if } z \in \mathbb{Z}^{d-1} \times \{0\} \\ \exp\left(a \cdot (z' - z)\right) \mu(z' - z) & \text{if } z \in \mathbb{Z}^{d-1} \times \mathbb{N}^* \end{cases} \end{aligned} \tag{8.4}$$

with $a = \overline{\hat{a}(q)}$. The infinite matrix $(\tilde{p}(z, z'),\ z, z' \in \mathbb{Z}^{d-1} \times \mathbb{N})$ is substochastic because $\varphi(\overline{\hat{a}(q)}) = 1$ and $\varphi_0(\overline{\hat{a}(q)}) \leq 1$ (see (8.3)). Green's function $\tilde{G}(z, z')$ of the twisted Markov process $\tilde{Z}(t)$ satisfies the equality

$$\tilde{G}(z, z') \;=\; G(z, z') \exp\Big(\overline{\hat{a}(q)} \cdot (z' - z)\Big), \qquad \forall z, z' \in \mathbb{Z}^{d-1} \times \mathbb{N} \tag{8.5}$$

and hence, using Proposition 5.6 we get

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{|z_n|} \log \tilde{G}(z, z_n) \;&=\; \overline{\hat{a}(q)} \cdot q + \liminf_{n\to\infty} \frac{1}{|z_n|} \log G(z, z_n) \\ &=\; \Big(\overline{\hat{a}(q)} - \hat{a}(q)\Big) \cdot q \;=\; 0 \end{aligned} \tag{8.6}$$

where the last relation holds because $q \in \mathbb{Z}^{d-1} \times \{0\}$ and the orthogonal projections of the points $\overline{\hat{a}(q)}$ and $\hat{a}(q)$ on the hyper-plane $\mathbb{R}^{d-1} \times \{0\}$ are identical according to the definition of the mapping $a \to \overline{a}$. Furthermore, we have to check that the twisted Markov-additive process $\tilde{Z}(t)$ satisfies the hypotheses (A1),(A2) and (A3) of Section 7. For this we first notice that the Markov process $Z(t)$ satisfies communication conditions (A1) because of Proposition 4.1 : for any $z, z' \in \mathbb{Z}^{d-1} \times \mathbb{N}^*$ there is a sequence of points $z_0, z_1, \ldots, z_n \in \mathbb{Z}^{d-1} \times \mathbb{N}^*$ with $z_0 = z$, $z_n = z'$ and $n \leq C|z' - z|$ such that and

$$|z_i - z_{i-1}| \leq C \quad \text{and} \quad \mathbb{P}_{z_{i-1}}(Z(1) = z_i) \;\geq\; \theta, \qquad \forall\, i = 1, \ldots, n.$$

For the twisted Markov process $\tilde{Z}(t)$ we have therefore

$$\mathbb{P}_{z_{i-1}}(\tilde{Z}(1) = z_i) \geq \exp\left(-\overline{\hat{a}(q)} \cdot (z_i - z_{i-1})\right)\theta \;\geq\; \exp\left(-C\left|\overline{\hat{a}(q)}\right|\right)\theta,$$

for all $i = 1, \dots, n$ and consequently, $\tilde{Z}(t)$ also satisfies communication condition (A1). Next, we remark that by Proposition 3.1, the constant multiples of the function $h_{\hat{a}(q)}$ are the only non-negative harmonic functions of the Markov process $Z(t)$ for which

$$\sup_{x \in \mathbb{R}^{d-1}} \exp(-\hat{\alpha}(q) \cdot x) h(x, y) \;<\; +\infty, \qquad \forall y \in \mathbb{N}$$

where $\hat{\alpha}(q)$ denotes the $d-1$ first coordinates of the point $\hat{a}(q)$. The constant multiples of the function

$$\tilde{h}(z) = \exp(-\overline{\hat{a}(q)} \cdot z) h_{\hat{a}(q)}(z)$$

are therefore the only non-negative harmonic functions of the twisted Markov process $\tilde{Z}(t)$ for which

$$\sup_{x \in \mathbb{R}^{d-1}} \tilde{h}(x, y) \;<\; +\infty \qquad \forall y \in \mathbb{N}.$$

Finally, the function

$$\sup_{z \in \mathbb{Z}^{d-1} \times \mathbb{N}} \mathbb{E}_z\big(\exp(a \cdot (\tilde{Z}(1) - z))\big) \;=\; \max\left\{\varphi\big(a + \overline{\hat{a}(q)}\big), \varphi_0\big(a + \overline{\hat{a}(q)}\big)\right\}$$

is finite everywhere on $\mathbb{R}^d$ because of the assumption (H4). The twisted Markov process $\tilde{Z}(t)$ satisfies therefore the hypotheses (A1),(A2) and (A3) of Section 7. Using Proposition 7.1 together with (8.6) we get

$$\lim_{n \to \infty} \tilde{G}(z, z_n)/\tilde{G}(z_0, z_n) = \tilde{h}(z)/\tilde{h}(z_0), \qquad \forall\, z \in \mathbb{Z}^{d-1} \times \mathbb{N}$$

and hence, using again (8.5) we obtain (8.1).

Case 2 : Suppose now that $q \in \mathbb{R}^{d-1} \times \mathbb{R}_+^*$ and $\varphi_0\left(\overline{\hat{a}(q)}\right) \;<\; 1$. Here, we apply Proposition 7.1 for a twisted Markov process $\tilde{Z}(t)$ having transition probabilities $\tilde{p}(z, z') \;=\; p(z, z') h_{\hat{a}(q)}(z')/h_{\hat{a}(q)}(z)$ and Green's function

$$\tilde{G}(z, z') = G(z, z') h_{\hat{a}(q)}(z')/h_{\hat{a}(q)}(z). \tag{8.7}$$

Such a Markov process is usually called h-transform of the original Markov process $Z(t)$. It is Markov-additive as well as the Markov process $Z(t)$ because the harmonic function $h_{\hat{a}(q)}$ satisfies the equality $h_{\hat{a}(q)}(x, y) \;=\; h_{\hat{a}(q)}(0, y)\exp(\hat{\alpha}(q) \cdot x)$ for all $(x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}$. Using quite the same arguments as in the previous case one can easily show that the new Markov-additive process $\tilde{Z}(t)$ satisfies the conditions (A1),(A2) and (A3) of Section 7. The last condition (A3) is satisfied here with the constant harmonic function $\tilde{h}(z) \equiv 1$. Moreover, from the explicit representation (1.10) of the harmonic function $h_{\hat{a}(q)}$ it follows that

$$\lim_{n \to \infty} \frac{1}{|z_n|} \log h_{\hat{a}(q)}(z_n) \;=\; \hat{a}(q) \cdot q$$

and hence, by Proposition 5.6,

$$\liminf_{n \to \infty} \frac{1}{|z_n|} \log \tilde{G}(z, z_n) \;=\; \hat{a}(q) \cdot q + \liminf_{n \to \infty} \frac{1}{|z_n|} \log G(z, z_n) \;=\; 0.$$

Using Proposition 7.1 we conclude therefore that

$$\lim_{n\to\infty} \tilde G(z,z_n)/\tilde G(z_0,z_n) = 1, \qquad \forall\, z\in\mathbb{Z}^{d-1}\times\mathbb{N}$$

and using next (8.7) we get (8.1).

Case 3 : Suppose now that $q\in\mathbb{R}^{d-1}\times\mathbb{R}_+^*$, $\varphi_0\left(\overline{\hat a(q)}\right) = 1$ and $\gamma_q\neq 0$. Recall that in this case,

$$h_{\hat a(q)}(z) \;=\; \exp\left(\overline{\hat a(q)}\cdot z\right), \qquad \forall\, z\in\mathbb{Z}^{d-1}\times\mathbb{N}. \tag{8.8}$$

Here, we can not use the above arguments because (8.6) does not hold and there is no harmonic function satisfying the equality (8.7). To prove (8.1) for such a vector $q\in\mathbb{R}^{d-1}\times\mathbb{R}_+^*$ we use Proposition 6.1 which proves that for any $\delta>0$ and $z\in\mathbb{Z}^{d-1}\times\mathbb{N}$,

$$G(z,z_n) \;\sim\; \Xi^q_\delta(z,z_n) \;\text{ as } n\to\infty \tag{8.9}$$

where

$$\Xi^q_\delta(z,z_n) \doteq \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\}:\ |w-\gamma_q|z_n||\leq|z_n|\delta,\\ w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*}} G(z,w)\mu_0(w')G_+(w+w',z_n) \tag{8.10}$$

In Case 1 , we have already proved that for all $z,z_0\in\mathbb{Z}^{d-1}\times\mathbb{N}$,

$$G(z,w)/G(z_0,w)\to h_{\hat a(\gamma)}(z)/h_{\hat a(\gamma)}(z')$$

when $|w|\to\infty$ and $w/|w|\to\gamma/|\gamma|\in\mathbb{R}^{d-1}\times\{0\}$. For any $\sigma>0$ there are therefore $n_\sigma>0$ and $\delta>0$ such that

$$(1-\sigma)h_{\hat a(\gamma_q)}(z)/h_{\hat a(\gamma_q)}(z_0) \;\leq\; G(z,w)/G(z_0,w) \;\leq\; (1+\sigma)h_{\hat a(q)}(z)/h_{\hat a(q)}(z_0)$$

whenever $|w-\gamma_q|z_n||<\delta|z_n|$ and $n>n_\sigma$. Using these inequalities in (8.10) we obtain

$$(1-\sigma)\frac{h_{\hat a(\gamma_q)}(z)}{h_{\hat a(\gamma_q)}(z_0)} \;\leq\; \frac{\Xi^q_\delta(z,z_n)}{\Xi^q_\delta(z_0,z_n)} \;\leq\; (1+\sigma)\frac{h_{\hat a(\gamma_q)}(z)}{h_{\hat a(\gamma_q)}(z_0)}.$$

for all $n>n_\sigma$. Next, letting $n\to\infty$ and using (8.9) we get

$$\begin{aligned}(1-\sigma)\frac{h_{\hat a(\gamma_q)}(z)}{h_{\hat a(\gamma_q)}(z_0)} \leq\; \liminf_{n\to\infty}\frac{\Xi^q_\delta(z,z_n)}{\Xi^q_\delta(z_0,z_n)} \;&=\; \liminf_{n\to\infty}\frac{G(z,z_n)}{G(z_0,z_n)}\\ \leq\; \limsup_{n\to\infty}\frac{G(z,z_n)}{G(z_0,z_n)} \;&=\; \limsup_{n\to\infty}\frac{\Xi^q_\delta(z,z_n)}{\Xi^q_\delta(z_0,z_n)} \;\leq\; (1+\sigma)\frac{h_{\hat a(\gamma_q)}(z)}{h_{\hat a(\gamma_q)}(z_0)}\end{aligned}$$

and finally, letting $\sigma\to 0$ we conclude that

$$\lim_{n\to\infty} G(z,z_n)/G(z_0,z_n) \;=\; h_{\hat a(\gamma_q)}(z)/h_{\hat a(\gamma_q)}(z_0).$$

The last relation combined with (8.2) proves (8.1).

Case 4 : Suppose finally that $q\in\mathbb{R}^{d-1}\times]0,+\infty[$, $\varphi_0\left(\overline{\hat a(q)}\right) = 1$ and $\gamma_q=0$. Here, the harmonic function $h_{\hat a(q)}$ is defined by (8.8). Since in this case $\varphi_0\left(\overline{\hat a(q)}\right) = \varphi\left(\overline{\hat a(q)}\right) = 1$ then without any restriction of generality we can assume that

$$\overline{\hat a(q)} = 0. \tag{8.11}$$

Otherwise, all the arguments below can be applied for the twisted Markov process having transition probabilities (8.4) with $a = \overline{\hat{a}(q)}$. So to get (8.1) we have to prove that

$$\lim_{n\to\infty} G(z,z_n)/G(z',z_n) \;=\; 1 \quad \text{for all } z,z' \in \mathbb{Z}^{d-1}\times\mathbb{N}. \tag{8.12}$$

We first prove this relation for the case when $z'-z \in \mathbb{Z}^{d-1}\times\{0\}$. For this we combine Proposition 6.1 and the results of Ignatiouk-Robert [15]. Recall that

$$G(z,z_n) \;=\; G_+(z,z_n) + \sum_{w\in\mathbb{Z}^{d-1}\times\{0\},w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*} G(z,w)\mu_0(w')G_+(w+w',z_n)$$

where $G_+(z,z')$ is Green's function of the homogeneous random walk $Z_+(t)$ on $\mathbb{Z}^{d-1}\times\mathbb{N}^*$ having transition probabilities $p(z,z') = \mu(z-z')$ which is killed upon hitting the boundary hyper-plane $\mathbb{Z}^{d-1}\times\{0\}$. By Proposition 6.1, when $n\to\infty$,

$$G(z,z_n) \sim \Xi^q_\delta(z,z_n) \doteq G_+(z,z_n) + \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\},w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*:\\ |w|\le\delta|z_n|}} G(z,w)\mu_0(w')G_+(w+w',z_n)$$

and for any $z' = z+u$ with $u\in\mathbb{Z}^{d-1}\times\{0\}$,

$$G(z',z_n) = G(z,z_n-u) \sim \Xi^q_\delta(z,z_n-u)$$

where

$$\Xi^q_\delta(z,z_n-u) \doteq G_+(z,z_n-u) + \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\},w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*:\\ |w|\le\delta|z_n|}} G(z,w)\mu_0(w')G_+(w+w',z_n-u)$$

$$= \; G_+(z+u,z_n) + \sum_{\substack{w\in\mathbb{Z}^{d-1}\times\{0\},w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*:\\ |w|\le\delta|z_n|}} G(z,w)\mu_0(w')G_+(w'+u,z_n-w) \tag{8.13}$$

Theorem 1 combined with Proposition 2.1 of Ignatiouk [15] proves that for all $w_0, w'' \in \mathbb{Z}^{d-1}\times\mathbb{N}^*$,

$$\frac{G_+(w'',v)}{G_+(w_0,v)} \to \frac{\exp(a(q)\cdot w'') - \exp(\overline{a(q)}\cdot w'')}{\exp(a(q)\cdot w_0) - exp(\overline{a(q)}\cdot w_0)}$$

as $|v|\to\infty$ and $v/|v| \to q \in \mathbb{R}^{d-1}\times]0,+\infty[$, $v\in\mathbb{Z}^{d-1}\times\mathbb{N}^*$. Recall that $a(q)$ denotes the unique point on the boundary ∂D of the set $D = \{a : \varphi(a)\le 1\}$ where the vector q is normal to D. In our case $q = q-\gamma_q$ and by Corollary 2.4, the vector $q-\gamma_q$ is normal to the set D at the point $\hat{a}(q)$. Hence $a(q) \;=\; \hat{a}(q)$ and according to our assumption (8.11),

$$\overline{a(q)} \;=\; \overline{\hat{a}(q)} \;=\; 0,$$

from which it follows that

$$G_+(w'',v)/G_+(w_0,v) \to (\exp(\hat{a}(q)\cdot w') - 1)/(\exp(\hat{a}(q)\cdot w_0) - 1)$$

as $|v|\to\infty$ and $v/|v|\to q \in \mathbb{R}^{d-1}\times]0,+\infty[$, $v\in\mathbb{Z}^{d-1}\times\mathbb{N}^*$. In particular, for $u\in\mathbb{Z}^{d-1}\times\{0\}$, from the definition of the mapping $a\to\overline{a}$ it follows that

$$\hat{a}(q)\cdot u = \overline{\hat{a}(q)}\cdot u = 0$$

and consequently,

$$\lim_{n\to\infty} \frac{G_+(z+u,z_n)}{G_+(z,z_n)} \;=\; 1, \qquad \forall z\in\mathbb{Z}^{d-1}\times\mathbb{N}^*. \tag{8.14}$$

Moreover, by Lemma 4.1 of Ignatiouk [15], the Markov process $(Z_+(t))$ satisfies communication condition on $\mathbb{Z}^{d-1}\times\mathbb{N}^*$: there exist $0<\theta<1$ and $C>0$ such that for any $w_0, w''\in\mathbb{Z}^{d-1}\times\mathbb{N}^*$ there is a sequence of points $w_1,\dots,w_n\in\mathbb{Z}^{d-1}\times\mathbb{N}^*$ with $w_n=w''$ and $n\leq C|w''-w_0|$ such that

$$|w_i-w_{i-1}|\leq C \quad\text{and}\quad \mu(w_i-w_{i-1})\geq\theta, \quad \forall\, i=1,\dots,n.$$

The probability that the Markov process $Z_+(t)$ starting at w_0 ever hits the point w'' is therefore greater than $\theta^n\geq\theta^{C|w_0-w''|}$ which implies that

$$G_+(w',v)/G_+(w_0,v)\leq\theta^{-C|w_0-w''|}$$

for all $v,w'',w_0\in\mathbb{Z}^{d-1}\times\mathbb{N}^*$. Since the exponential functions are integrable with respect to the probability measure μ_0, by dominated convergence theorem from this it follows that

(8.15)
$$\sum_{w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*}\mu_0(w')\frac{G_+(w'+u,v)}{G_+(w_0,v)} \;\to\; \sum_{w'\in\mathbb{Z}^{d-1}\times\mathbb{N}^*}\mu_0(w')\frac{\exp(\hat a(q)\cdot(w'+u))-1}{\exp(\hat a(q)\cdot w_0)-1}$$

as $|v|\to\infty$ and $v/|v|\to q\in\mathbb{R}^{d-1}\times]0,+\infty[$, $v\in\mathbb{Z}^{d-1}\times\mathbb{N}^*$. Remark finally that the right hand side of the above display is equal to

$$\sum_{w'\in\mathbb{Z}^{d-1}\times\mathbb{N}}\mu_0(w')\frac{\exp(\hat a(q)\cdot w')-1}{\exp(\hat a(q)\cdot w_0)-1} \;=\; \frac{\varphi_0(\hat a(q))-1}{\exp(\hat a(q)\cdot w_0)-1}$$

because $\overline{\hat a(q)}=0$ and according to the definition of the mapping $a\to\overline{a}$,

$$\hat a(q)\cdot w=\overline{\hat a(q)}\cdot w$$

for all $w\in\mathbb{Z}^{d-1}\times\{0\}$. Using therefore (8.14) and (8.15) with $v=z_n-w$ for the right hand side of (8.13) we obtain

$$\Xi^q_\delta(z,z_n-u)\;\sim\; G_+(z,z_n)+\sum_{w\in\mathbb{Z}^{d-1}\times\{0\}:\ |w|\leq\delta|z_n|}G(z,w)G_+(w_0,z_n-w)\frac{\varphi_0(\hat a(q))-1}{\exp(\hat a(q)\cdot w_0)-1}$$

when $n\to\infty$ and $\delta\to0$. Since the right hand side of the last display does not depend on $u\in\mathbb{Z}^{d-1}\times\{0\}$ this proves that

$$\lim_{\delta\to0}\liminf_{n\to\infty}\frac{\Xi^q_\delta(z,z_n-u)}{\Xi^q_\delta(z,z_n)} \;=\; \lim_{\delta\to0}\limsup_{n\to\infty}\frac{\Xi^q_\delta(z,z_n-u)}{\Xi^q_\delta(z,z_n)} \;=\; 1$$

for all $u\in\mathbb{Z}^{d-1}\times\{0\}$. The equality (8.12) for $z\in\mathbb{Z}^{d-1}\times\mathbb{N}$ and $z'=z+u$ with $u\in\mathbb{Z}^{d-1}\times\{0\}$ follows now from Proposition 6.1.

Next, we prove (8.12) for arbitrary $z,z'\in\mathbb{Z}^{d-1}\times\mathbb{N}$. Recall that by Proposition 4.1, the Markov process $(Z(t))$ satisfies communication condition on $\mathbb{Z}^{d-1}\times\mathbb{N}$ and consequently, there are $0<\delta<1$ and $C>0$ such that for any $z,z'\in\mathbb{Z}^{d-1}\times\mathbb{N}$, the probability that the Markov process $(Z(t))$ starting at z ever hits the point z' is greater than $\theta^{C|z-z'|}$. From this it follows that

$$\theta^{C|z-z'|}\;\leq\; G(z,z_n)/G(z',z_n)\;\leq\;\theta^{-C|z-z'|}$$

for all $z,z'\in\mathbb{Z}^{d-1}\times\mathbb{N}$ and $n\in\mathbb{N}$. Since under the hypotheses (H4), the exponential functions are integrable with respect to the probability measures μ and μ_0, by

dominated convergence theorem we conclude that for any sub-sequence n_k for which the sequence of functions

$$K_n(z) \ = \ G(z, z_{n_k})/G(z_0, z_{n_k})$$

converge point-wise, the limit

$$K(z) \doteq \lim_{k\to\infty} K_{n_k}(z) \ \geq \ e^{-\theta|z-z_0|}$$

is a harmonic function for $(Z(t))$. Remark now that $K(z_0) = 1$ and

$$K(z+u) \ = \ K(z) \qquad \forall\, z \in \mathbb{Z}^{d-1}\times\mathbb{N},\ u \in \mathbb{Z}^{d-1}\times\{0\} \tag{8.16}$$

because (8.12) is already proved for $z' = z + u$ with $u \in \mathbb{Z}^{d-1}\times\{0\}$. This implies that $K(z) = 1$ for all $z \in \mathbb{Z}^{d-1}\times\mathbb{N}$ because by Proposition 3.1, the only non-negative harmonic functions satisfying the equality (8.16) are the constant multiples of the function $h_{\hat{a}(q)}(z)$. These arguments prove that the sequence of functions K_n converge point-wise to the function K because the function K does not depend on the sub-sequence n_k. The equality (8.12) is therefore proved.

References

[1] Patrick Billingsley, *Convergence of probability measures*, Wiley series in probability and mathematical statistics, John Wiley & Sons Ltd, New York, 1968.

[2] V. M. Blinovskiĭ and R. L. Dobrushin, *Process level large deviations for a class of piecewise homogeneous random walks*, The Dynkin Festschrift Progr. Probab., vol. 34, Birkhäuser Boston, Boston, MA, 1994, pp. 1–59.

[3] A.A. Borovkov, *On Cramer transform, large deviations in boundary-value problems and conditional invariance principle*, Sib.Math.J **3** (1995), 493–509.

[4] A.A. Borovkov and A.A Mogulskii, *Large deviations for Markov chains in the positive quadrant*, Russian Math. Surveys **56** (2001), no. 5, 803–916.

[5] Amir Dembo and Ofer Zeitouni, *Large deviations techniques and applications*, Springer-Verlag, New York, 1998.

[6] Robert D.Foley and David R.McDonald, *Bridges and networks: exact asymptotics*, Ann. Appl. Probab. **15** (2005), no. 1B, 542–586.

[7] J. L. Doob, J. L. Snell, and R. E. Williamson, *Application of boundary theory to sums of independent random variables.*, Contributions to probability and statistics, Stanford Univ. Press, Stanford, Calif., 1960, pp. 182–197.

[8] P. Dupuis, R. Ellis, and A. Weiss, *Large deviations for Markov processes with discontinuous statistics I : General upper bounds*, Annals of Probability **19** (1991), no. 3, 1280–1297.

[9] Paul Dupuis and Richard S. Ellis, *Large deviations for Markov processes with discontinuous statistics. II. Random walks*, Probability Theory and Related Fields **91** (1992), no. 2, 153–194.

[10] ———, *The large deviation principle for a general class of queueing systems. I*, Transactions of the American Mathematical Society **347** (1995), no. 8, 2689–2751.

[11] M. I. Freidlin and A. D. Wentzell, *Random perturbations of dynamical systems*, second ed., Springer-Verlag, New York, 1998, Translated from the 1979 Russian original by Joseph Szücs.

[12] P.L. Hennequin, *Processus de Markoff en cascade*, Ann. Inst. H. Poincaré **18** (1963), no. 2, 109–196.

[13] Irina Ignatiouk-Robert, *Sample path large deviations and convergence parameters*, Annals of Applied Probability **11** (2001), no. 4, 1292–1329.

[14] ———, *Large deviations for processes with discontinuous statistics*, 2005, pp. 1479–1508.

[15] ———, *Martin boundary of a killed random walk on a half-space*, Journal of Theoretical Probability, **21** 2008, no. 1, 35-68.

[16] I. A. Ignatyuk, V. A. Malyshev, and V. V. Shcherbakov, *The influence of boundaries in problems on large deviations*, Uspekhi Matematicheskikh Nauk **49** (1994), no. 2(296), 43–102.

[17] I.A. Kurkova and V.A. Malyshev, *Martin boundary and elliptic curves.*, Markov Processes Related Fields **4** (1998), 203–272.

[18] P. Ney and Spitzer F, *The martin boundary for random walk*, Trans. Amer. Math. Soc. (1966), no. 121, 116–132.
[19] R. Tyrrell Rockafellar, *Convex analysis*, Princeton University Press, Princeton, NJ, 1997, Reprint of the 1970 original, Princeton Paperbacks.
[20] Philippe Robert, *Stochastic Metworks and Queues*, Springer-Verlag, Berlin, 2003.
[21] E. Seneta, *Nonnegative matrices and Markov chains*, second ed., Springer-Verlag, New York, 1981.
[22] A. Shwartz and A. Weiss, *Large deviations for performance analysis*, Stochastic Modeling Series, Chapman & Hall, London, London, 1995.
[23] Wolfgang Woess, *Random walks on infinite graphs and groups*, Cambridge University Press, Cambridge, 2000.

Université de Cergy-Pontoise, Département de mathématiques, 2, Avenue Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France
E-mail address: Irina.Ignatiouk@math.u-cergy.fr